

# A numerical method for the solution of integral equations of Mellin type<sup>☆</sup>

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## Abstract

We are interested in the numerical solution of second kind integral equations of Mellin convolution type. We describe a modified Nyström method based on the Gauss-Lobatto or Gauss-Radau quadrature rule. Under certain assumptions on the Mellin kernel, we prove the stability and the convergence of the proposed procedure and also derive error estimates. Finally, some test problems are solved and the numerical results showing the effectiveness of our method are presented.

*Keywords:* Gaussian rule, Mellin kernel, integral equations of Mellin type, Nyström method  
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## 1. Introduction

We are interested in the numerical solution of second kind integral equations of the following type

$$f(y) + \int_0^1 [K(x, y) + H(x, y)]f(x)dx = g(y), \quad 0 < y \leq 1, \quad (1)$$

where  $K(x, y)$  is a Mellin kernel, continuous for all  $x + y > 0$  and such that

$$K(x, y) = \pm \frac{1}{x} k\left(\frac{y}{x}\right), \quad (2)$$

for some given function  $k : [0, \infty) \rightarrow [0, \infty)$  satisfying the following assumption

$$\int_0^\infty \frac{k(t)}{t} dt < \infty, \quad (3)$$

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<sup>☆</sup>Dedicated to Professor Francesco A. Costabile on the occasion of his 70th birthday  
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$H(x, y)$  and  $g(y)$  are given continuous functions, and  $f(x)$  is the unknown.

The mathematical formulation of many problems in physics and engineering leads to solve integral equations of the form (1). For instance, they occur when boundary integral methods are applied in order to solve potential problems on planar domains with corners or crack problems in linear elasticity (see [1, 3, 16] and the references therein).

Representing the solution of the interior Dirichlet problem on a domain with piecewise smooth boundary in the form of a double layer potential leads to a system of integral equations involving Mellin convolution operators with kernels  $K(x, y)$  of the form

$$K(x, y) = \frac{1}{\pi} \frac{y \sin(\chi\pi)}{x^2 + 2xy \cos(\chi\pi) + y^2} \quad (4)$$

with  $(1 - \chi)\pi$ ,  $\chi \in (-1, 1)$ , the interior angle at the corner point (see [2, 11]). The kernel

$$K(x, y) = \frac{4}{\pi} \frac{xy^2}{(x^2 + y^2)^2} \quad (5)$$

arises from a problem of determining the distribution of stress in a thin elastic plate in the vicinity of a cruciform crack (see [16, 14, 17]).

In general the solution  $f(y)$  of (1) (or perhaps its higher derivatives) will have a singularity at  $y = 0$ . However, when proposing a numerical method for it, the main difficulty one encounters is the proof of its stability, since, while the integral operators

$$(\mathcal{K}f)(y) = \int_0^1 K(x, y)f(x)dx \quad (6)$$

and

$$(\mathcal{H}f)(y) = \int_0^1 H(x, y)f(x)dx \quad (7)$$

are both bounded maps on the space  $C[0, 1]$ , only the operator  $\mathcal{H}$  is compact. The Mellin convolution operator  $\mathcal{K}$  is not compact being its kernel  $K(x, y)$  not smooth or weakly singular on  $[0, 1] \times [0, 1]$ , but containing a fixed first order singularity at  $x = y = 0$ .

When we are interested in the numerical solution of equation (1) by means of Nyström or discrete collocation methods, the evaluation of the integral transform of Mellin type  $\mathcal{K}f$  at some chosen collocation points represents a crucial step. Hence efficient quadrature formulas are necessary in order to approximate the integrals  $(\mathcal{K}f)(y)$ ,  $y \in (0, 1]$ .

In this paper, at first, we propose an algorithm for the evaluation of such integrals, since the fixed singularity of the Mellin kernel  $K(x, y)$  at the origin could make inefficient the use of the classical Gaussian rules when  $y$  is very close to the endpoint 0. Then, following [14, 15], we propose a “modified” Nyström type method for the numerical solution of (1).

In [14] the Authors propose to modify the Gauss-Legendre or a suitable product quadrature rule in order to construct a stable Nyström type approximant for

the solution of a particular convolution equation, whose Mellin kernel is given by (5) and whose right hand side and solution vanish at the origin.

In [15] a Nyström method for the solution of more general Mellin convolution integral equations, i.e. equations of type (1), with  $K(x, y) = \pm \frac{1}{x} k\left(\frac{y}{x}\right)$  satisfying (3) and  $H(x, y) \equiv 0$ , is proposed. It is based on a product quadrature formula constructed by replacing the function  $f(x)$  by its Lagrange polynomial associated with the nodes of the Gauss-Radau quadrature rule. The rule is modified near the origin in order to prove the stability of the Nyström interpolant. The case of solutions non vanishing at the origin is also considered here. The proposed procedure can be applied once one has determined the value  $f(0)$ , and this can be easily done if the integral in (3) is known.

Our method is based on a slight modification of the classical Gauss-Radau or Gauss-Lobatto rules. These modified quadrature formulas give rise to stable and convergent procedures for the numerical solution of Mellin integral equations of type (1), which can be employed in the more general case where the solution  $f(x)$  does not necessarily vanish at the origin and  $H(x, y) \neq 0$ .

Let us remark that when  $f(0) \neq 0$  it is not possible to use the modified Gauss-Legendre rule considered in [14] as well as when  $H(x, y) \neq 0$  the approach described in [15] cannot be followed. The stability and convergence analysis performed in [15] does not apply anymore. In such a case only the Radau and Lobatto ones can be used. In this way, since one of the quadrature nodes coincides with the interval endpoint 0, the value  $f(0)$  turns out to be one of the final linear system unknowns. We also prove that this linear system is well conditioned.

Very recently a different modified Nyström type method has been proposed in [4] for the numerical treatment of integral equations having the form (1), but with the Mellin kernel  $K(x, y)$  in (2) satisfying the condition

$$\int_0^\infty \frac{k(t)}{t^{\frac{1}{2}}} dt < \infty \quad (8)$$

instead of (3). In this case the Mellin operator  $\mathcal{K}$  defined in (6) is a continuous map from  $L^2[0, 1]$  into itself but it could be not bounded with respect to the uniform norm. As a consequence, the solution  $f$  of (1) could be singular at the origin. Therefore, the Gauss Legendre quadrature rule is used and a different modification near the singularity point is performed in order to achieve stability for the numerical procedure.

Let us observe that we can write

$$(\mathcal{K}f)(y) = \int_0^1 \frac{1}{x} k\left(\frac{y}{x}\right) f(x) dx = \int_0^\infty \frac{1}{t} k\left(\frac{1}{t}\right) \chi_{[0, 1/y]}(t) f(yt) dt,$$

where  $\chi_{[0, 1/y]}$  denotes the characteristic function on the interval  $[0, 1/y]$ . Then, taking into account (3), by applying the Lebesgue convergence theorem, we further have

$$\lim_{y \rightarrow 0} (\mathcal{K}f)(y) = \lim_{y \rightarrow 0} \int_0^\infty \frac{1}{t} k\left(\frac{1}{t}\right) \chi_{[0, 1/y]}(t) f(yt) dt = f(0) \left[ \int_0^\infty \frac{k(t)}{t} dt \right]. \quad (9)$$

Consequently, under the assumption (3), we can extend the definition of the function  $(\mathcal{K}f)(y)$  in the point  $y = 0$  as follows

$$(\mathcal{K}f)(y) = \begin{cases} \int_0^1 K(x, y)f(x)dx, & y \in (0, 1] \\ f(0) \left[ \int_0^\infty \frac{k(t)}{t} dt \right], & y = 0. \end{cases} \quad (10)$$

Moreover, the following theorem establishes that the operator  $\mathcal{K}$  is linear and bounded on the Banach space  $(C[0, 1], \|\cdot\|_\infty)$  into itself.

**Theorem 1.1.** *Assuming the kernel  $K(x, y)$  as in (2) with  $k$  satisfying (3), one has that the operator  $\mathcal{K}$  defined by (10) is a bounded linear operator from  $C[0, 1]$  into  $C[0, 1]$  with*

$$\|\mathcal{K}\|_\infty = \int_0^\infty \frac{k(t)}{t} dt. \quad (11)$$

## 2. Preliminaries

### 2.1. Functions spaces

For  $1 \leq p < \infty$ , let us define the space  $L^p = L^p[0, 1]$  as the set of all measurable functions such that

$$\|f\|_p = \left( \int_0^1 |f(x)|^p dx \right)^{\frac{1}{p}} < \infty$$

and, for  $p = \infty$ , we set  $L^\infty = C[0, 1]$ , equipped with the norm

$$\|f\|_\infty = \max_{0 \leq x \leq 1} |f(x)|.$$

For  $1 \leq p \leq \infty$ ,  $l \in \mathbb{N}_0$ ,  $\sigma \in \mathbb{R}$ , we consider the weighted Sobolev spaces

$$X_\sigma^{p,l} = \left\{ f : x^{j-\sigma} f^{(j)}(x) \in L^p, \ j = 0, \dots, l \right\},$$

endowed with the norm

$$\|f\|_{p,l,\sigma} = \sum_{j=0}^l \|x^{j-\sigma} f^{(j)}\|_p.$$

Let us also introduce the Sobolev-type subspace  $W_r^p$  of  $L^p$  defined as follows

$$W_r^p = \{f \in L^p : f^{(r-1)} \in AC(0, 1) \text{ and } \|f^{(r)} \varphi^r\|_p < \infty\},$$

equipped with the norm

$$\|f\|_{W_r^p} = \|f\|_p + \|f^{(r)} \varphi^r\|_p,$$

where  $r \in \mathbb{N}$ ,  $\varphi(x) = \sqrt{x(1-x)}$  and  $AC(0, 1)$  is the set of all functions that are absolutely continuous in every compact subset of  $(0, 1)$ .

## 2.2. Quadrature rules

The numerical method we are going to propose will be based on the following quadrature rule on the interval  $[0, 1]$

$$\int_0^1 f(x)dx = \sum_{i=1}^m \lambda_{m,i} f(x_{m,i}) + e_m(f), \quad (12)$$

where  $\lambda_{m,i}$  and  $x_{m,i}$  are the Christoffel numbers and the nodes, respectively, associated with the Gauss, Gauss-Radau or Gauss-Lobatto formula with respect to the Legendre weight, and  $e_m$  denotes the quadrature error.

In order to give an estimate for  $e_m(f)$ , which will be useful in the sequel, we recall the definition of the weighted error of best polynomial approximation

$$E_m(f)_{w,p} = \inf_{P_m \in \mathbb{P}_m} \|(f - P_m)w\|_p,$$

where  $1 \leq p \leq \infty$ ,  $w$  denotes a general weight function on  $[0, 1]$ , and  $\mathbb{P}_m$  is the set of all algebraic polynomials on  $[0, 1]$  of degree at most  $m$ .

Moreover, in the following  $\mathcal{C}$  denotes a positive constant which may takes different values on different occurrences. We write  $\mathcal{C} = \mathcal{C}(a, b, \dots)$  to say that  $\mathcal{C}$  is dependent on the parameters  $a, b, \dots$  and  $\mathcal{C} \neq \mathcal{C}(a, b, \dots)$  to say that  $\mathcal{C}$  is independent of them.

**Theorem 2.1.** [13] *For all  $f \in W_r^1$ ,  $r \in \mathbb{N}$ , it results*

$$|e_m(f)| \leq \frac{\mathcal{C}}{m^r} E_{2m-1-r} \left( f^{(r)} \right)_{\varphi^r, 1}, \quad (13)$$

where  $\mathcal{C} \neq \mathcal{C}(m, f)$ .

## 3. The modified quadrature rule and its error estimates

Let us define the following finite rank operator

$$(\mathcal{K}_m f)(y) = \sum_{i=1}^m \lambda_{m,i} K(x_{m,i}, y) f(x_{m,i}), \quad (14)$$

associated with the integral operator  $\mathcal{K}$  defined in (10) and the quadrature rule (12). Since the kernel  $K(x, y)$  is not continuous on  $[0, 1] \times [0, 1]$ , the approximating operators  $\mathcal{K}_m$  are not pointwise convergent to  $\mathcal{K}$  on the space  $C[0, 1]$ . Therefore the standard analysis for the classical Nyström method, based on (14), do not apply.

In order to establish the stability and the convergence of our numerical method, we need some additional assumptions on the Mellin kernel  $K(x, y)$ .

Let us assume that  $K(x, y)$  possesses partial derivatives with respect to  $x$  up to a certain order  $r \in \mathbb{N}$  and there exist  $r$  constants  $C_j$ ,  $j = 1, \dots, r$ , such that

$$y^{j/2} \left\| \frac{\partial^j}{\partial x^j} K(\cdot, y) \varphi^j \right\|_1 \leq C_j, \quad y \in (0, 1], \quad j = 1, \dots, r. \quad (15)$$

For instance, the kernels (4) and (5) satisfy (15) for any  $r \in \mathbb{N}$ .

Following an idea in [14, 15], we propose to approximate the integral operator  $\mathcal{K}$  on  $C[0, 1]$ , rather than by the operator  $\mathcal{K}_m$  in (14), by a new operator  $\tilde{\mathcal{K}}_m$ , which is a slight modification of it.

Setting  $y_m = \frac{c}{m^{2-2\epsilon}}$ , for some fixed positive constant  $c$  and arbitrarily small  $\epsilon > 0$ , we define the “modified” operator  $\tilde{\mathcal{K}}_m$  as follows

$$(\tilde{\mathcal{K}}_m f)(y) = \begin{cases} (\mathcal{K}_m f)(y), & y \in [y_m, 1] \\ \frac{1}{y_m} [y (\mathcal{K}_m f)(y_m) + (y_m - y) (\mathcal{K} f)(0)], & y \in [0, y_m] \end{cases}. \quad (16)$$

In other words, on the interval  $[0, y_m)$  the function  $(\tilde{\mathcal{K}}_m f)(y)$  approximating  $(\mathcal{K} f)(y)$  is the linear polynomial assuming the values  $(\mathcal{K} f)(0)$  and  $(\mathcal{K}_m f)(y_m)$  at the interpolation points 0 and  $y_m$ , respectively. Let us remark that the definition of the “modified” operator given above is a bit different from that one given in [14, 15] and coincides with it only in the special case when  $f(0) = 0$ .

The operators  $\tilde{\mathcal{K}}_m$  defined above satisfy the following crucial theorem.

**Theorem 3.1.** *Let us assume that the assumptions of Theorem 1.1 and condition (15) are satisfied. Then the operators  $\tilde{\mathcal{K}}_m$  defined by (16) are linear maps from  $C[0, 1]$  into  $C[0, 1]$  such that*

$$\lim_{m \rightarrow \infty} \|\tilde{\mathcal{K}}_m\|_\infty \leq \int_0^\infty \frac{k(t)}{t} dt \quad (17)$$

and

$$\lim_{m \rightarrow \infty} \|(\tilde{\mathcal{K}}_m - \mathcal{K})f\|_\infty = 0, \quad \forall f \in C[0, 1], \quad (18)$$

with the operator  $\mathcal{K}$  defined by (10).

Moreover, we can provide, under suitable assumptions both on the Mellin kernel  $K(x, y)$  and on the function  $f$ , a pointwise estimate for the error  $|(\tilde{\mathcal{K}}_m - \mathcal{K})f(y)|$ . To this end, we need to suppose that the function  $k(t)$  in (2) satisfies the condition

$$\int_0^\infty t^{-1-\sigma} |t^j k^{(j)}(t)| dt < \infty, \quad j = 0, \dots, l, \quad (19)$$

for some  $\sigma > 0$  and  $l \in \mathbb{N}_0$ , and the function  $f$  is of the form

$$f(x) = x^\sigma f_0(x) + f_1(x), \quad (20)$$

with  $f_0 \in C^l(0, 1]$  such that

$$x^j f_0^{(j)}(x) \in C[0, 1], \quad j = 0, \dots, l \quad (21)$$

and  $f_1$  a smoother function in  $[0, 1]$ . Let us observe that, in this case,  $f \in X_\sigma^{\infty, l}$ . Moreover, if  $l \geq 1$  and  $\sigma > 1/2$  then  $f \in W_r^\infty$  with  $r \geq \min\{l, \lfloor 2\sigma \rfloor\}$  ( $\lfloor 2\sigma \rfloor$  denotes the integer part of  $2\sigma$ ).

**Theorem 3.2.** Assume that the kernel  $K(x, y)$  in (2) verifies (15), for some  $r \in \mathbb{N}$ , and (19), for some  $\sigma > 0$  and  $l \geq 1$ . Moreover, let us suppose that the function  $f$  takes the form (20) with  $f_0$  satisfying (21) and  $f_1$  a smoother function in  $[0, 1]$ . If  $f \in W_r^\infty$ , the following pointwise error estimate

$$|(\tilde{\mathcal{K}}_m - \mathcal{K})f(y)| \leq \begin{cases} \mathcal{C} \max \left\{ \frac{1}{m^r} y y_m^{-r/2-1}, y y_m^{\sigma-1}, y^\sigma \right\}, & y \in [0, y_m) \\ \frac{\mathcal{C}}{m^r} y^{-r/2}, & y \in [y_m, 1] \end{cases} \quad (22)$$

holds true, with  $\mathcal{C} \neq \mathcal{C}(m, y)$ .

From (22) we can deduce that, for any fixed  $y \in [0, 1]$ , the error tends to zero when  $m$  goes to infinity. Moreover, we emphasize that the approximation error depends on the evaluation point  $y \in [0, 1]$ . More precisely, for a fixed  $m$ , in the interval  $[y_m, 1]$  the error decreases for increasing values of  $y$ , while in  $[0, y_m)$  it becomes smaller and smaller as well as  $y$  is closer to the endpoint 0.

Let us also remark that from the previous error estimates it follows that the error is of order  $O(m^{-r})$  in any interval of the type  $[\delta, 1]$ , for any fixed  $0 < \delta < 1$ , and at most of order  $O(m^{-2\sigma+\gamma})$ , with  $\gamma$  arbitrarily small, in the whole interval  $[0, 1]$ , when  $r$  is arbitrarily large.

#### 4. The modified Nyström method

The modified quadrature rule described in Section 3 has a natural application in the numerical solution of integral equations of type (1), characterized by the presence of non-compact Mellin integral operators.

Defined the operators  $\mathcal{K}$  and  $\mathcal{H}$  as in (6) and (7), respectively, as a consequence of the noncompactness of the operator  $\mathcal{K}$ , the standard theory for the analysis of the equation

$$(\mathcal{I} + \mathcal{K} + \mathcal{H})f = g, \quad (23)$$

as well as of the classical procedures for its numerical solution, cannot be applied. Nevertheless, if the operator  $(\mathcal{I} + \mathcal{K})$  is invertible and  $\text{Ker}(\mathcal{I} + \mathcal{K} + \mathcal{H}) = \{0\}$ , then also the operator  $\mathcal{I} + \mathcal{K} + \mathcal{H}$  has a bounded inverse. For instance, this holds true when

$$\|\mathcal{K}\|_\infty < 1. \quad (24)$$

Numerical methods for solving (1) have been proposed by several authors (see [16, 11, 14, 15, 6, 7, 9, 12] and the references therein). Many of them are based on piecewise polynomial approximation with graded meshes, sometimes combined with suitable smoothing transformations of the variables. Moreover, some modification techniques have been applied to collocation and quadrature methods based on piecewise or global approximation, which allow to prove stability and convergence results.

In this paper we propose a Nyström-type method which only makes use of the “modified” quadrature formula described above, without resorting to any change of variables.

Throughout this section we shall assume that the relation (24) is fulfilled and the kernels of the integral operators  $\mathcal{K}$  and  $\mathcal{H}$  satisfy the following conditions.

For the Mellin kernel  $K(x, y) = \pm \frac{1}{x} k\left(\frac{y}{x}\right)$ , as in the previous section, we shall suppose that, for some  $r \in \mathbb{N}$ , there exist  $r$  constants  $C_j$ ,  $j = 1, \dots, r$ , such that

$$y^{j/2} \left\| \frac{\partial^j}{\partial x^j} K(\cdot, y) \varphi^j \right\|_1 \leq C_j, \quad y \in (0, 1], \quad j = 1, \dots, r, \quad (25)$$

and, for some  $\sigma > 0$  and  $l \in \mathbb{N}$ , the relations

$$\int_0^\infty t^{-1-\sigma} |t^j k^{(j)}(t)| dt < \infty, \quad j = 0, \dots, l, \quad (26)$$

are fulfilled. Moreover, we will also make the additional assumption

$$k \in X_\sigma^{\infty, l-1}. \quad (27)$$

For the smoother kernel  $H(x, y)$ , we shall suppose that

$$\sup_{y \in [0, 1]} \|H(\cdot, y)\|_{W_r^\infty} < +\infty. \quad (28)$$

Then, we can assume that the solution  $f$  takes the form

$$f(x) = x^\sigma f_0(x) + f_1(x) \quad (29)$$

with  $f_0$  satisfying (21) and  $f_1$  is a smoother function such that  $x^{-\sigma} f_1 \in C[0, 1]$  at the least. This behavior of  $f$  can be deduced from [8, Theorem 1.10], where it is established that under certain hypotheses on the right-hand side  $g$ , the solution  $f$  belongs to the space  $X_\sigma^{\infty, l}$ .

The “modified” Nyström method consists in solving, in place of (23), the approximating equation

$$(\mathcal{I} + \tilde{\mathcal{K}}_m + \mathcal{H}_m) \tilde{f}_m(y) = g(y) \quad (30)$$

where the operator  $\tilde{\mathcal{K}}_m$ , defined by (16), is based on the Gauss-Radau or Gauss-Lobatto rule with respect to the Legendre weight and  $\mathcal{H}_m$  is given by

$$(\mathcal{H}_m f)(y) = \sum_{i=1}^m \lambda_{m,i} H(x_{m,i}, y) f(x_{m,i}), \quad (31)$$

with  $\lambda_{m,i}$  and  $x_{m,i}$ ,  $i = 1, \dots, m$ , the weights and the nodes of the same quadrature formula.

By collocating equation (30) at the quadrature nodes  $x_{m,i}$ ,  $i = 1, \dots, m$ , we reduce to solve the following linear system

$$(\mathcal{I} + \tilde{\mathcal{K}}_m + \mathcal{H}_m) \tilde{f}_m(x_{m,i}) = g(x_{m,i}), \quad i = 1, \dots, m, \quad (32)$$



in the unknowns  $\tilde{f}_m(x_{m,i})$ ,  $i = 1, \dots, m$ . The solution of system (32) permits us to construct the Nyström interpolant

$$\tilde{f}_m(y) = g(y) - (\tilde{\mathcal{K}}_m + \mathcal{H}_m)\tilde{f}_m(y), \quad (33)$$

solution of (30). Viceversa, each solution  $\tilde{f}_m$  of (30) furnishes a solution of system (32). It will merely be sufficient to evaluate  $\tilde{f}_m$  at the nodes of the quadrature formula.

The following theorem establishes the stability and convergence of the proposed method.

**Theorem 4.1.** *Assume that  $\text{Ker}(\mathcal{I} + \mathcal{K} + \mathcal{H}) = \{0\}$  in the space  $C[0, 1]$  and the conditions (24)-(28) are satisfied. Then, for sufficiently large  $m$ , say  $m \geq m_0$ , the operators  $\mathcal{I} + \tilde{\mathcal{K}}_m + \mathcal{H}_m$  are invertible and their inverses are uniformly bounded on  $C[0, 1]$ . Moreover, assuming that the solution  $f$  of (23) has the form (29), with  $f_0$  satisfying (21) and  $f_1$  a smoother function such that  $x^{-\sigma} f_1 \in C[0, 1]$ , if  $f \in W_r^\infty$ , the solution  $\tilde{f}_m$  of equation (30) satisfies the following error estimate*

$$\|f - \tilde{f}_m\|_\infty \leq \mathcal{C} \left[ \|(\mathcal{K} - \tilde{\mathcal{K}}_m)f\|_\infty + \|(\mathcal{H} - \mathcal{H}_m)f\|_\infty \right], \quad \mathcal{C} \neq \mathcal{C}(m), \quad (34)$$

where  $\|(\mathcal{K} - \tilde{\mathcal{K}}_m)f\|_\infty$  can be estimated taking into account (22) and

$$\|(\mathcal{H} - \mathcal{H}_m)f\|_\infty \leq \frac{\mathcal{C}}{m^r}, \quad \mathcal{C} \neq \mathcal{C}(m). \quad (35)$$

We can also prove the following result concerning the conditioning of the linear system (32).

**Theorem 4.2.** *Denoting by  $\text{cond}(\mathcal{I} + \tilde{\mathcal{K}}_m + \mathcal{H}_m)$  the condition number of the operator  $\mathcal{I} + \tilde{\mathcal{K}}_m + \mathcal{H}_m : C[0, 1] \rightarrow C[0, 1]$ , by  $A_m$  the matrix associated to the linear system (32) and by  $\text{cond}(A_m)$  its condition number in infinity norm, we have, for any  $m \geq m_0$*

$$\text{cond}(A_m) \leq \text{cond}(\mathcal{I} + \tilde{\mathcal{K}}_m + \mathcal{H}_m) \leq \mathcal{C}, \quad (36)$$

where  $\mathcal{C} \neq \mathcal{C}(m)$ .

From the previous results we can deduce that the described procedure is stable, convergent and it leads to solve well conditioned linear systems. The performance of the method is illustrated by the numerical examples shown in Section 6.

## 5. Proofs

*Proof of Theorem 1.1.* The linearity of the operator  $\mathcal{K}$  is obvious from (10). For any  $f \in C[0, 1]$ , the continuity of  $(\mathcal{K}f)(y)$  can be trivially proved for all  $y \in (0, 1]$ , since  $K(x, y)$  is continuous for  $x + y > 0$ . The continuity at  $y = 0$

is a consequence of definition (10), taking also into account (9). It remains to prove (11). Let  $f \in C[0, 1]$  be such that  $\|f\|_\infty = 1$ . Then, for any  $y \in (0, 1]$ , we can write

$$\begin{aligned} |(\mathcal{K}f)(y)| &\leq \int_0^1 |K(x, y)| dx = \int_0^1 \frac{1}{x} k\left(\frac{y}{x}\right) dx \\ &= \int_0^{\frac{1}{y}} \frac{1}{s} k\left(\frac{1}{s}\right) ds = \int_y^\infty \frac{k(t)}{t} dt \leq \int_0^\infty \frac{k(t)}{t} dt, \end{aligned}$$

and for  $y = 0$

$$|(\mathcal{K}f)(0)| \leq \int_0^\infty \frac{k(t)}{t} dt, \quad (37)$$

which is a trivial consequence of the definition (10). In particular, when  $f \equiv 1$ , we have

$$\|\mathcal{K}f\|_\infty = \int_0^\infty \frac{k(t)}{t} dt.$$

Then, (11) is proved and, in virtue of (3), we can assert that the operator  $\mathcal{K}$  is bounded on  $C[0, 1]$ .  $\square$

*Proof of Theorem 3.1.* The linearity of the operator  $\tilde{\mathcal{K}}_m$  is a trivial consequence of the linearity of both  $\mathcal{K}$  and  $\mathcal{K}_m$ . Let us prove, now, that, for every function  $f \in C[0, 1]$ ,  $\tilde{\mathcal{K}}_m f \in C[0, 1]$ . Since  $K(x, y)$  is continuous for  $x + y > 0$  the continuity of  $\tilde{\mathcal{K}}_m f$  for all  $y \in (y_m, 1]$  is obvious. When  $y \in [0, y_m)$ ,  $(\tilde{\mathcal{K}}_m f)(y)$  is also a continuous function being, by definition, a linear polynomial. It remains to show that  $\tilde{\mathcal{K}}_m f$  is continuous at the point  $y = y_m$ . But this is easily seen, being

$$\lim_{y \rightarrow y_m^-} (\tilde{\mathcal{K}}_m f)(y) = (\mathcal{K}_m f)(y_m) = \lim_{y \rightarrow y_m^+} (\tilde{\mathcal{K}}_m f)(y).$$

In order to prove (17), let us consider  $f \in C[0, 1]$  such that  $\|f\|_\infty \leq 1$ . One has

$$\|\tilde{\mathcal{K}}_m f\|_\infty = \max \left\{ \sup_{y \in [0, y_m]} |(\tilde{\mathcal{K}}_m f)(y)|, \sup_{y \in [y_m, 1]} |(\mathcal{K}_m f)(y)| \right\}. \quad (38)$$

We start by estimating the second term into the braces. For  $y \in [y_m, 1]$ , we can write

$$\begin{aligned} |(\mathcal{K}_m f)(y)| &\leq \|f\|_\infty \sum_{i=1}^m \lambda_{m,i} |K(x_{m,i}, y)| \leq \left| \sum_{i=1}^m \lambda_{m,i} K(x_{m,i}, y) \right| \\ &= \left| \int_0^1 K(x, y) dx - e_m(K(\cdot, y)) \right| \leq \int_0^1 |K(x, y)| dx + |e_m(K(\cdot, y))| \\ &\leq \sup_{y \in (0, 1]} \int_0^1 \frac{1}{x} k\left(\frac{y}{x}\right) dx + \sup_{y \in [y_m, 1]} |e_m(K(\cdot, y))| \\ &\leq \int_0^\infty \frac{k(t)}{t} dt + \sup_{y \in [y_m, 1]} |e_m(K(\cdot, y))|. \end{aligned}$$

Now, using the error estimate (13) for the quadrature rule (12) and taking into account the assumption (15) on the kernel, we have, for  $y \in [y_m, 1]$ ,

$$\begin{aligned} |e_m(K(\cdot, y))| &\leq \frac{\mathcal{C}}{m^r} E_{2m-1-r} \left( \frac{\partial^r}{\partial x^r} K(\cdot, y) \right)_{\varphi^r, 1} \leq \frac{\mathcal{C}}{m^r} \left\| \frac{\partial^r}{\partial x^r} K(\cdot, y) \varphi^r \right\|_1 \\ &\leq \frac{\mathcal{C}}{m^r} y^{-r/2} \leq \frac{\mathcal{C}}{m^{r\epsilon}}, \quad \mathcal{C} \neq \mathcal{C}(m), \end{aligned}$$

and, consequently,

$$\sup_{y \in [y_m, 1]} |(\mathcal{K}_m f)(y)| \leq \int_0^\infty \frac{k(t)}{t} + \frac{\mathcal{C}}{m^{r\epsilon}}. \quad (39)$$

Let us estimate the quantity  $\sup_{y \in [0, y_m)} |(\tilde{\mathcal{K}}_m f)(y)|$ , in order to complete the proof of (17). For  $y \in [0, y_m)$ , we can write

$$\begin{aligned} |(\tilde{\mathcal{K}}_m f)(y)| &\leq \frac{1}{y_m} \sup_{y \in [0, y_m)} [y |(\mathcal{K}_m f)(y_m)| + (y_m - y) |(\mathcal{K} f)(0)|] \\ &= \max \{ |(\mathcal{K}_m f)(y_m)|, |(\mathcal{K} f)(0)| \} \\ &\leq \max \left\{ \sup_{y \in [y_m, 1]} |(\mathcal{K}_m f)(y)|, |(\mathcal{K} f)(0)| \right\} \end{aligned}$$

from which, also according to (37), we deduce

$$\sup_{y \in [0, y_m)} |(\tilde{\mathcal{K}}_m f)(y)| \leq \int_0^\infty \frac{k(t)}{t} + \frac{\mathcal{C}}{m^{r\epsilon}}. \quad (40)$$

Finally, combining (38), (39) and (40) we have that

$$\|\tilde{\mathcal{K}}_m f\|_\infty \leq \int_0^\infty \frac{k(t)}{t} + \frac{\mathcal{C}}{m^{r\epsilon}}, \quad (41)$$

and (17) follows.

In order to prove (18), we want to apply the Banach-Steinhaus theorem (see, for instance, [1, p. 517]), recalling that the set  $\mathbb{P}$  of all algebraic polynomials on  $[0, 1]$  is a dense subspace of  $C[0, 1]$ . To this end, we are interested in showing that the operators  $\tilde{\mathcal{K}}_m : C[0, 1] \rightarrow C[0, 1]$  are uniformly bounded with respect to  $m$ , i.e.

$$\sup_m \|\tilde{\mathcal{K}}_m\|_\infty < \infty \quad (42)$$

and pointwise convergent to  $\mathcal{K}$  on the subspace  $\mathbb{P}$ , i.e.

$$\lim_{m \rightarrow \infty} \|(\tilde{\mathcal{K}}_m - \mathcal{K})p\|_\infty = 0, \quad \forall p \in \mathbb{P}. \quad (43)$$

Assertion (42) follows from (41). Let us show (43), noting that

$$\|(\tilde{\mathcal{K}}_m - \mathcal{K})p\|_\infty = \max \left\{ \sup_{y \in [0, y_m)} |(\tilde{\mathcal{K}}_m - \mathcal{K})p(y)|, \sup_{y \in [y_m, 1]} |(\tilde{\mathcal{K}}_m - \mathcal{K})p(y)| \right\},$$

and, then, proving that both the terms into the braces converges to zero when  $m \rightarrow \infty$ . Fixed  $p \in \mathbb{P}$ , by using the error estimate (13) for the quadrature formula (12) applied to the function  $K(\cdot, y)p$ , we have

$$|(\tilde{\mathcal{K}}_m - \mathcal{K})p(y)| \leq \frac{\mathcal{C}}{m^r} \int_0^1 \left| \frac{\partial^r}{\partial x^r} [K(x, y)p(x)] \right| \varphi^r(x) dx. \quad (44)$$

But, taking into account the assumption (15), we get

$$\begin{aligned} \int_0^1 \left| \frac{\partial^r}{\partial x^r} [K(x, y)p(x)] \right| \varphi^r(x) dx &= \int_0^1 \left| \sum_{j=0}^r \binom{r}{j} \frac{\partial^j}{\partial x^j} K(x, y) p^{(r-j)}(x) \right| \varphi^r(x) dx \\ &\leq \sum_{j=0}^r \binom{r}{j} \int_0^1 \left| \frac{\partial^j}{\partial x^j} K(x, y) \right| \varphi^j(x) |p^{(r-j)}(x)| \varphi^{r-j}(x) dx \\ &\leq \mathcal{C} \sum_{j=0}^r \binom{r}{j} \int_0^1 \left| \frac{\partial^j}{\partial x^j} K(x, y) \right| \varphi^j(x) dx \leq \mathcal{C} \sum_{j=0}^r \binom{r}{j} y^{-j/2} \leq \mathcal{C} y^{-r/2} \end{aligned}$$

with  $\mathcal{C} = \mathcal{C}(r, p)$ , from which it follows (for sufficiently large  $m$ )

$$\sup_{y \in [y_m, 1]} \int_0^1 \left| \frac{\partial^r}{\partial x^r} [K(x, y)p(x)] \right| \varphi^r(x) dx \leq \mathcal{C} m^{r-r\epsilon}.$$

Combining this result with (44), we obtain

$$\lim_{m \rightarrow \infty} \sup_{y \in [y_m, 1]} |(\tilde{\mathcal{K}}_m - \mathcal{K})p(y)| = 0. \quad (45)$$

Now, let us consider  $y \in [0, y_m)$ . By the definition (16), we have

$$\begin{aligned} |(\tilde{\mathcal{K}}_m - \mathcal{K})p(y)| &= \left| \frac{1}{y_m} [y (\mathcal{K}_m p)(y_m) + (y_m - y) (\mathcal{K} p)(0)] - (\mathcal{K} p)(y) \right| \\ &\leq \frac{y}{y_m} [|(\mathcal{K}_m p)(y_m) - (\mathcal{K} p)(y_m)| + |(\mathcal{K} p)(y_m) - (\mathcal{K} p)(0)|] + |(\mathcal{K} p)(0) - (\mathcal{K} p)(y)|. \end{aligned}$$

We observe that

$$\begin{aligned} \sup_{y \in [0, y_m)} \frac{1}{y_m} y |(\mathcal{K}_m p)(y_m) - (\mathcal{K} p)(y_m)| &= |(\mathcal{K}_m p)(y_m) - (\mathcal{K} p)(y_m)| \\ &\leq \sup_{y \in [y_m, 1]} |(\tilde{\mathcal{K}}_m - \mathcal{K})p(y)|, \end{aligned}$$

and from (45) it follows that

$$\lim_{m \rightarrow \infty} \sup_{y \in [0, y_m)} \frac{1}{y_m} y |(\mathcal{K}_m p)(y_m) - (\mathcal{K} p)(y_m)| = 0. \quad (46)$$

Moreover, we can write

$$\sup_{y \in [0, y_m)} \frac{1}{y_m} y |(\mathcal{K} p)(y_m) - (\mathcal{K} p)(0)| = |(\mathcal{K} p)(y_m) - (\mathcal{K} p)(0)|$$

and, then,

$$\lim_{m \rightarrow \infty} \sup_{y \in [0, y_m)} \frac{1}{y_m} y |(\mathcal{K} p)(y_m) - (\mathcal{K} p)(0)| = 0, \quad (47)$$

since  $\mathcal{K} p \in C[0, 1]$  and  $\lim_{m \rightarrow \infty} y_m = 0$ . For the same reasons, we also have

$$\lim_{m \rightarrow \infty} \sup_{y \in [0, y_m)} |(\mathcal{K} p)(0) - (\mathcal{K} p)(y)| = 0. \quad (48)$$

Finally taking into account (46)-(48), we obtain

$$\lim_{m \rightarrow \infty} \sup_{y \in [0, y_m)} |(\tilde{\mathcal{K}}_m - \mathcal{K})p(y)| = 0,$$

and the proof is complete.  $\square$

*Proof of Theorem 3.2.* Let us consider separately the cases  $y \in [0, y_m)$  and  $y \in [y_m, 1]$ . For  $y \in [y_m, 1]$ , proceeding as in the proof of Theorem 3.1, taking into account (15) and using [5, p. 310] we can write

$$\begin{aligned} |(\tilde{\mathcal{K}}_m - \mathcal{K})f(y)| &\leq \frac{\mathcal{C}}{m^r} \int_0^1 \left| \frac{\partial^r}{\partial x^r} [K(x, y)f(x)] \right| \varphi^r(x) dx \\ &\leq \frac{\mathcal{C}}{m^r} \sum_{j=0}^r \binom{r}{j} \int_0^1 \left| \frac{\partial^j}{\partial x^j} K(x, y) \right| \varphi^j(x) \left| f^{(r-j)}(x) \right| \varphi^{r-j}(x) dx \\ &\leq \frac{\mathcal{C}}{m^r} \|f\|_{W_r^\infty} \sum_{j=0}^r \binom{r}{j} y^{-j/2} \leq \frac{\mathcal{C}}{m^r} y^{-r/2} \end{aligned}$$

with  $\mathcal{C} = \mathcal{C}(r, f)$ . In the case when  $y \in [0, y_m)$  we have

$$\begin{aligned} |(\tilde{\mathcal{K}}_m - \mathcal{K})f(y)| &\leq \frac{y}{y_m} |(\mathcal{K}_m f)(y_m) - (\mathcal{K} f)(y_m)| \\ &\quad + \frac{y}{y_m} |(\mathcal{K} f)(y_m) - (\mathcal{K} f)(0)| + |(\mathcal{K} f)(0) - (\mathcal{K} f)(y)| \\ &=: A_1(y) + A_2(y) + A_3(y). \end{aligned}$$

From the previous estimate, for  $A_1(y)$ , we can deduce that

$$A_1(y) \leq \frac{\mathcal{C}}{m^r} y y_m^{-r/2-1}.$$

In order to estimate the quantity  $A_2(y)$ , let us observe that

$$\begin{aligned} A_2(y) &= \frac{y}{y_m} \left| \int_0^1 \frac{1}{x} k\left(\frac{y_m}{x}\right) f(x) dx - f(0) \int_0^\infty \frac{k(t)}{t} dt \right| \\ &\leq \frac{y}{y_m} \left| \int_0^1 \frac{1}{x} k\left(\frac{y_m}{x}\right) [f(x) - f(0)] dx \right| \\ &\quad + \frac{y}{y_m} \left| f(0) \left[ \int_0^1 \frac{1}{x} k\left(\frac{y_m}{x}\right) dx - \int_0^\infty \frac{k(t)}{t} dt \right] \right| =: B_1(y) + B_2(y). \end{aligned}$$

Then, we can reduce to estimate  $B_1(y)$  and  $B_2(y)$ , just defined. By the assumptions we have

$$\begin{aligned} B_1(y) &\leq \frac{y}{y_m} y_m^\sigma \int_0^{\frac{1}{y_m}} t^{\sigma-1} k\left(\frac{1}{t}\right) |f_0(y_m t)| dt \\ &\quad + \frac{y}{y_m} \int_0^{\frac{1}{y_m}} \frac{1}{t} k\left(\frac{1}{t}\right) |f_1(y_m t) - f_1(0)| dt =: B_{1,1}(y) + B_{1,2}(y). \end{aligned}$$

For the first addendum, taking into account the hypothesis (19), we can write

$$\begin{aligned} B_{1,1}(y) &\leq \|f_0\|_\infty y y_m^{\sigma-1} \int_0^{\frac{1}{y_m}} t^{\sigma-1} k\left(\frac{1}{t}\right) dt \leq \|f_0\|_\infty y y_m^{\sigma-1} \int_{y_m}^\infty t^{-1-\sigma} k(t) dt \\ &\leq \|f_0\|_\infty y y_m^{\sigma-1} \int_0^\infty t^{-1-\sigma} k(t) dt \leq C y y_m^{\sigma-1}. \end{aligned}$$

In order to estimate  $B_{1,2}(y)$ , first, we observe that for any  $t \in \left[0, \frac{1}{y_m}\right]$  there exists a point  $\xi_{m,t} \in [0, y_m t]$  such that

$$|f_1(\xi_{m,t})| = \max_{x \in [0, y_m t]} |f_1(x)|.$$

Then, we have that

$$\begin{aligned} B_{1,2}(y) &\leq 2y y_m^{\sigma-1} \int_0^{\frac{1}{y_m}} t^{\sigma-1} k\left(\frac{1}{t}\right) (y_m t)^{-\sigma} |f_1(\xi_{m,t})| dt \\ &\leq 2y y_m^{\sigma-1} \int_0^{\frac{1}{y_m}} t^{\sigma-1} k\left(\frac{1}{t}\right) (\xi_{m,t})^{-\sigma} |f_1(\xi_{m,t})| dt \\ &\leq C y y_m^{\sigma-1} \int_{y_m}^\infty t^{-1-\sigma} k(t) dt \leq C y y_m^{\sigma-1}, \end{aligned}$$

since, from the assumptions,  $x^{-\sigma} f_1(x) \in C[0, 1]$ . Consequently, we can conclude that

$$B_1(y) \leq C y y_m^{\sigma-1}.$$

For  $B_2(y)$ , by (19), we can write

$$\begin{aligned} B_2(y) &= \frac{y}{y_m} |f(0)| \left| \int_{y_m}^\infty \frac{k(t)}{t} dt - \int_0^\infty \frac{k(t)}{t} dt \right| \\ &= \frac{y}{y_m} |f(0)| \int_0^{y_m} t^{-1-\sigma} k(t) t^\sigma dt \leq C y y_m^{\sigma-1}, \end{aligned}$$

and, then, deduce

$$A_2(y) \leq C y y_m^{\sigma-1}.$$

By proceeding in an analogous way, for  $A_3(y)$  we can prove that

$$\begin{aligned} A_3(y) &\leq \left| \int_0^1 \frac{1}{x} k\left(\frac{y}{x}\right) [f(x) - f(0)] dx \right| \\ &+ \left| f(0) \left[ \int_0^1 \frac{1}{x} k\left(\frac{y}{x}\right) dx - \int_0^\infty \frac{k(t)}{t} dt \right] \right| \leq C y^\sigma. \end{aligned}$$

Finally, by collecting all the previous estimates, we obtain the thesis.  $\square$

*Proof of Theorem 4.1.* First, we observe that, in virtue of (42) and (18), we can deduce that the operators  $\mathcal{I} + \tilde{\mathcal{K}}_m : C[0, 1] \rightarrow C[0, 1]$  are bounded and pointwise convergent to  $\mathcal{I} + \tilde{\mathcal{K}}$ . Moreover, from (17), (11) and (24), in virtue of the geometric series theorem, it follows that, for sufficiently large  $m$ , say  $m \geq m_0$ , the operators  $(\mathcal{I} + \tilde{\mathcal{K}}_m)^{-1} : C[0, 1] \rightarrow C[0, 1]$  exist and are uniformly bounded with

$$\|(\mathcal{I} + \tilde{\mathcal{K}}_m)^{-1}\| \leq \frac{1}{1 - \sup_{m \geq m_0} \|\tilde{\mathcal{K}}_m\|}.$$

Now, taking into account that the sequence  $\{\mathcal{H}_m\}_m$  is collectively compact and pointwise convergent to the integral operator  $\mathcal{H}$  (see, for instance, [1]), it results (see, for instance, Theorem 10.8 and Problem 10.3 in [10]) that for, sufficiently large  $m$ , the operators  $(\mathcal{I} + \tilde{\mathcal{K}}_m + \mathcal{H}_m)^{-1}$  exist and are uniformly bounded, i.e. the method is stable. From this, since

$$f - \tilde{f}_m = (\mathcal{I} + \tilde{\mathcal{K}}_m + \mathcal{H}_m)^{-1} \left[ (\tilde{\mathcal{K}}_m - \mathcal{K})f + (\mathcal{H}_m - \mathcal{H})f \right],$$

we immediately deduce (34). In order to complete the proof it remains to show the estimate (35). By using (13), from the assumptions we can deduce

$$\begin{aligned} |(\mathcal{H} - \mathcal{H}_m)f(y)| &\leq \frac{\mathcal{C}}{m^r} \int_0^1 \left| \frac{\partial^r}{\partial x^r} [H(x, y)f(x)] \right| \varphi^r(x) dx \\ &\leq \frac{\mathcal{C}}{m^r} \sum_{j=0}^r \binom{r}{j} \int_0^1 \left| \frac{\partial^j H(x, y)}{\partial x^j} \right| \varphi^j(x) |f^{(r-j)}(x)| \varphi^{r-j}(x) dx \\ &\leq \frac{\mathcal{C}}{m^r} \|f\|_{W_r^\infty} \|H(\cdot, y)\|_{W_r^2} \leq \frac{\mathcal{C}}{m^r} \|f\|_{W_r^2}, \end{aligned}$$

with  $\mathcal{C} \neq C(m, y)$ , from which (35) immediately follows.  $\square$

*Proof of Theorem 4.2.* The proof can be conducted by proving, following a standard scheme (see [1]), both the inequality

$$\|A_m\|_\infty \leq \|\mathcal{I} + \tilde{\mathcal{K}}_m + \mathcal{H}_m\|_\infty$$

and

$$\|A_m^{-1}\|_\infty \leq \|(\mathcal{I} + \tilde{\mathcal{K}}_m + \mathcal{H}_m)^{-1}\|_\infty,$$

from which the thesis immediately follows.  $\square$

## 6. Numerical tests

In this section we show the numerical results obtained by applying the proposed modified Nyström method for solving some integral equations of type (1). In all the examples we specify the values of the parameters  $c$  and  $\epsilon$  involved in the definition of the breaking point  $y_m = \frac{c}{m^{2-2\epsilon}}$ . Since the Gaussian quadrature formulas  $(\mathcal{K}_m f)(y)$  in (14) suffers a loss of accuracy when the point  $y$  is very close to  $y = 0$ , especially for larger values of  $r$  (see (22)), these values will be chosen in order to ensure that the point  $y_m$  is sufficiently far from the singularity point. More precisely, we take into account the behaviour that the solution  $f$  has near the origin (see (29)), when condition (26) is fulfilled for some  $\sigma > 0$  and the right-hand side  $g$  is a sufficiently smooth function. In this case we propose to take  $c = \gamma \cdot 10^{2\sigma}$ , with  $0 < \gamma < 1$  and the small quantity  $\epsilon$  such that  $\epsilon \leq 10^{-1}$ .

In the tables that follow we report the pointwise absolute errors

$$\tilde{e}_m(y) = |f(y) - \tilde{f}_m(y)|, \quad y \in (0, 1], \quad (49)$$

where  $f$  is the exact solution of (1) and  $\tilde{f}_m$  is the solution of the equation (30), given by (33). When the exact solution  $f$  is not known we shall consider as exact the approximating one  $\tilde{f}_{2048}$ . We also show the values of the condition number in infinity norm of the matrix  $A_m$  associated with the linear system (32).

Finally, for each  $m$  we precise the number  $j$  of the quadrature nodes which fall in the interval  $[0, y_m)$ .

**Example 6.1.** *Let us consider the second kind Mellin integral equation of type (1) with*

$$K(x, y) = -\frac{1}{\pi} \frac{y \sin(\pi/4)}{y^2 + 2xy \cos(\pi/4) + y^2}, \quad H(x, y) = (x + y) \cos(x + y)$$

*and the right-hand side function  $g(x)$  obtained by setting  $f(x) = x^3 - 1$ . Let us observe that for this equation the condition (24) is fulfilled as well as (25) and (28) hold true for any  $r \in \mathbb{N}$ . Moreover, since the Mellin kernel  $K(x, y)$  can be represented in the form (2) with*

$$k(t) = -\frac{1}{\pi} \frac{t \sin(\pi/4)}{1 + 2t \cos(\pi/4) + t^2},$$

*the assumptions (26) and (27) are satisfied for any  $l \in \mathbb{N}$ , provided that  $\sigma < 1$ . In Table 1 we report the errors*

$$e_m(y) = |f(y) - f_m(y)|,$$

*where  $f_m$  is the interpolant constructed by applying the classical Nyström method based on the Gauss-Legendre quadrature rule. They show that such procedure is inefficient when the evaluation point  $y$  is very close to the endpoint 0.*



Table 1: Ex. 6.1, Gauss-Legendre

$m$	$e_m(0.9)$	$e_m(0.5)$	$e_m(0.1)$	$e_m(10^{-2})$	$e_m(10^{-6})$	$e_m(10^{-10})$
$2^3$	6.04e-04	7.00e-05	2.17e-03	7.30e-02	2.49e-01	2.50e-01
$2^4$	2.06e-04	1.93e-05	8.68e-04	6.52e-03	2.49e-01	2.50e-01
$2^5$	6.94e-05	6.30e-06	2.98e-04	1.87e-03	2.49e-01	2.50e-01
$2^6$	2.31e-05	2.09e-06	9.99e-05	6.71e-04	2.48e-01	2.50e-01
$2^7$	7.69e-06	6.95e-07	3.31e-05	2.25e-04	2.42e-01	2.49e-01
$2^8$	2.54e-06	2.30e-07	1.09e-05	7.46e-05	2.22e-01	2.49e-01
$2^9$	8.40e-07	7.60e-08	3.62e-06	2.46e-05	1.57e-01	2.49e-01
$2^{10}$	2.77e-07	2.50e-08	1.19e-06	8.13e-06	4.39e-02	2.49e-01

As shown in Table 2, the performance of our method is better than the one of the classical Nyström method, also when the evaluation point  $y$  is distant from the fixed singularity. In Table 3 are reported the condition numbers of the matrix associated with the linear system (32), which confirm the theoretical result (36).

Table 2: Ex. 6.1,  $c = 0.5 \cdot 10^{2 \cdot 0.9}$ ,  $\epsilon = 10^{-3}$ , Lobatto-Legendre

$m$	$j$	$\tilde{e}_m(0.9)$	$\tilde{e}_m(0.5)$	$\tilde{e}_m(0.1)$	$\tilde{e}_m(10^{-2})$	$\tilde{e}_m(10^{-6})$	$\tilde{e}_m(10^{-10})$
$2^3$	4	1.22e-03	9.30e-04	8.44e-03	5.20e-04	7.52e-04	7.52e-04
$2^4$	4	5.01e-05	1.14e-05	8.87e-04	4.42e-04	6.47e-07	6.96e-07
$2^5$	4	1.24e-06	1.39e-07	4.74e-06	9.48e-05	9.21e-08	7.80e-08
$2^6$	4	2.75e-08	2.56e-09	1.16e-07	5.92e-07	5.70e-09	1.99e-09
$2^7$	4	5.84e-10	5.30e-11	2.51e-09	1.61e-08	9.87e-10	4.31e-11
$2^8$	4	1.25e-11	1.13e-12	5.43e-11	3.66e-10	2.40e-10	9.52e-13
$2^9$	4	3.81e-13	3.59e-14	1.66e-12	1.13e-11	7.01e-11	3.44e-14
$2^{10}$	4	4.24e-14	5.10e-15	2.02e-13	1.39e-12	5.69e-11	8.21e-15

**Example 6.2.** Let us consider the second kind Mellin integral equation of type (1) with

$$K(x, y) = \frac{2}{\pi} \frac{x^2 y}{(x^2 + y^2)^2}, \quad H(x, y) = (y + 3) \exp(x), \quad g(x) = x^3 \exp(x + 1).$$

The kernels satisfy the assumptions as in Example 6.1. Comparing the errors reported in Table 4 with the ones given in Table 6, we deduce that, also in this case, our method outperforms the classical Nyström method based on the Gauss-Legendre quadrature rule. We also remark that our results have been obtained by solving well conditioned linear systems as shown in Table 5.

Table 3: Ex. 6.1,  $c = 0.5 \cdot 10^{2 \cdot 0.9}$ ,  $\epsilon = 10^{-3}$ , Lobatto-Legendre

$m$	$\text{cond}(A_m)$
$2^3$	2.8433
$2^4$	2.8254
$2^5$	2.8499
$2^6$	2.8696
$2^7$	2.8793
$2^8$	2.8835
$2^9$	2.8853
$2^{10}$	2.8860

Table 4: Ex. 6.2,  $c = 0.8 \cdot 10^{2 \cdot 0.95}$ ,  $\epsilon = 10^{-3}$ , Radau-Legendre

$m$	$j$	$\tilde{e}_m(0.9)$	$\tilde{e}_m(0.5)$	$\tilde{e}_m(0.1)$	$\tilde{e}_m(10^{-2})$	$\tilde{e}_m(10^{-6})$	$\tilde{e}_m(10^{-10})$
$2^3$	7	6.19e-02	1.12e-01	5.86e-02	7.07e-02	6.05e-02	6.05e-02
$2^4$	6	3.95e-03	3.38e-03	3.98e-02	1.23e-02	2.19e-03	2.19e-03
$2^5$	5	2.60e-04	2.08e-04	3.33e-04	7.13e-03	1.45e-04	1.46e-04
$2^6$	5	6.11e-06	4.86e-06	3.74e-06	1.01e-03	3.20e-06	3.43e-06
$2^7$	5	6.45e-08	5.13e-08	3.74e-08	2.58e-07	2.46e-08	3.62e-08
$2^8$	5	6.88e-10	5.47e-10	3.99e-10	4.30e-10	1.76e-08	3.84e-10
$2^9$	5	4.47e-11	3.56e-11	2.59e-11	2.51e-11	9.05e-09	2.42e-11
$2^{10}$	5	9.48e-13	7.62e-13	5.57e-13	5.24e-13	2.12e-09	3.25e-13

Table 5: Ex. 6.2,  $c = 0.8 \cdot 10^{2 \cdot 0.95}$ ,  $\epsilon = 10^{-3}$ , Radau-Legendre

$m$	$\text{cond}(A_m)$
$2^3$	1.5881e+01
$2^4$	1.5414e+01
$2^5$	1.5436e+01
$2^6$	1.5447e+01
$2^7$	1.5450e+01
$2^8$	1.5451e+01
$2^9$	1.5451e+01
$2^{10}$	1.5451e+01

Table 6: Ex. 6.2, Gauss-Legendre

$m$	$\tilde{e}_m(0.9)$	$\tilde{e}_m(0.5)$	$\tilde{e}_m(0.1)$	$\tilde{e}_m(10^{-2})$	$\tilde{e}_m(10^{-6})$	$\tilde{e}_m(10^{-10})$
$2^3$	2.29e-03	1.83e-03	8.92e-04	8.61e-02	4.85e-01	2.44e-03
$2^4$	6.17e-04	4.91e-04	3.72e-04	3.13e-02	4.83e-01	1.02e-03
$2^5$	1.59e-04	1.27e-04	9.26e-05	5.74e-04	4.82e-01	6.41e-04
$2^6$	4.05e-05	3.22e-05	2.35e-05	2.76e-05	4.78e-01	5.40e-04
$2^7$	1.01e-05	8.10e-06	5.90e-06	5.73e-06	4.63e-01	5.13e-04
$2^8$	2.52e-06	2.01e-06	1.46e-06	1.41e-06	4.04e-01	5.01e-04
$2^9$	6.02e-07	4.79e-07	3.49e-07	3.38e-07	1.83e-01	4.75e-04
$2^{10}$	1.20e-07	9.60e-08	6.99e-08	6.77e-08	1.43e-01	3.80e-04

Table 7: Ex. 6.3,  $c = 0.01 \cdot 10^{2 \cdot 1.9}$ ,  $\epsilon = 10^{-3}$ , Radau-Legendre

$m$	$j$	$\tilde{e}_m(0.9)$	$\tilde{e}_m(0.5)$	$\tilde{e}_m(0.1)$	$\tilde{e}_m(10^{-2})$	$\tilde{e}_m(10^{-6})$	$\tilde{e}_m(10^{-10})$
$2^3$	7	1.97e-04	9.76e-03	1.77e-02	3.18e-03	5.42e-04	5.42e-04
$2^4$	6	5.79e-04	1.58e-03	1.04e-02	1.08e-03	9.79e-04	9.79e-04
$2^5$	5	1.06e-05	2.01e-05	9.50e-05	4.65e-04	1.76e-05	1.77e-05
$2^6$	5	1.27e-07	2.28e-07	2.78e-07	3.79e-05	1.94e-07	2.04e-07
$2^7$	5	1.76e-09	3.18e-09	2.98e-09	6.62e-08	4.18e-10	2.83e-09
$2^8$	5	2.86e-11	5.17e-11	4.99e-11	5.68e-11	4.97e-10	4.60e-11
$2^9$	5	5.00e-13	9.07e-13	8.80e-13	6.59e-13	6.33e-11	8.06e-13
$2^{10}$	5	1.48e-14	2.73e-14	3.00e-14	2.87e-14	4.00e-11	3.30e-14

**Example 6.3.** We consider the integral equation (1) with

$$K(x, y) = \frac{4}{\pi} \frac{xy^2}{(x^2 + y^2)^2}, \quad H(x, y) = \frac{\exp(x + y + 2)}{x^2 + y^2 + 1}, \quad g(x) = x^2(x + \cos(x)).$$

It is easy to verify that (24) holds true and (25) is fulfilled for any  $r \in \mathbb{N}$ . Furthermore, since the Mellin kernel can be represented in the form (2) with

$$k(t) = \frac{4}{\pi} \frac{t^2}{(1 + t^2)^2},$$

the assumptions (26) and (27) are satisfied for any  $l \in \mathbb{N}$ , provided that  $\sigma < 2$ . The numerical results obtained by applying the proposed procedure are shown in tables 7-8.

We remark that the numerical evidence shows that choices of the parameters  $\gamma$  and  $\epsilon$  different from those reported in the numerical examples, as long as

Table 8: Ex. 6.3,  $c = 0.01 \cdot 10^{2 \cdot 1.9}$ ,  $\epsilon = 10^{-3}$ , Radau-Legendre

$m$	$\text{cond}(A_m)$
$2^3$	3.172e+01
$2^4$	3.066e+01
$2^5$	3.124e+01
$2^6$	3.138e+01
$2^7$	3.142e+01
$2^8$	3.142e+01
$2^9$	3.143e+01
$2^{10}$	3.143e+01

$0 < \gamma < 1$  and  $\epsilon \leq 10^{-1}$ , do not produce significant changes in the accuracy of the results and, in the above tests, we have on purpose taken three different values of  $\gamma$ .

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### References

- [1] K.E. Atkinson, The Numerical Solution of Integral Equations of the Second Kind, volume 552 of *Cambridge Monographs on Applied and Computational Mathematics*, Cambridge Univ. Press, Cambridge, 1997.
- [2] K.E. Atkinson, F.R. de Hoog, The numerical solution of laplace's equation on a wedge, *IMA J. Numer. Anal.* 4 (1984) 19–41.
- [3] G.A. Chandler, I.G. Graham, Product integration collocation methods for non-compact integral operator equations, *Math. Comp.* 50 (1988) 125–138.
- [4] M.C. De Bonis, C. Laurita, A modified Nyström method for integral equations with Mellin type kernels, *J. Comp. Appl. Math.* 296 (2016) 512–527.
- [5] Z. Ditzian, On interpolation of  $L_p[a, b]$  and weighted Sobolev spaces, *Pacific J. Math.* 90 (1980) 307–323.
- [6] D. Elliott, The cruciform crack problem and sigmoidal transformations, *Math. Methods Appl. Sci.* 20 (1997) 121–132.
- [7] D. Elliott, S. Prössdorf, An algorithm for the approximate solution of integral equations of Mellin type, *Numer. Math.* 70 (1995) 427–452.

- [8] J. Elschner, On spline approximation for a class of non-compact integral equations, *Math. Nachr.* 146 (1990) 271–321.
- [9] J. Elschner, I.G. Graham, Numerical methods for integral equations of Mellin type, *J. Comput. Appl. Math.* 125 (2000) 423–437.
- [10] R. Kress, Linear Integral Equations, volume 82 of *Applied Mathematical Sciences*, Springer-Verlag, Berlin, 1989.
- [11] R. Kress, A Nyström method for boundary integral equations in domains with corners, *Numer. Math.* 58 (1990) 445–461.
- [12] G. Mastroianni, C. Frammartino, A. Rathsfeld, On polynomial collocation for second kind integral equations with fixed singularities of Mellin type, *Numer. Math.* 94 (2003) 333–365.
- [13] G. Mastroianni, G.V. Milovanovic, Interpolation processes. Basic theory and applications, Springer Monographs in Mathematics, Springer Verlag, Berlin, 2008.
- [14] G. Mastroianni, G. Monegato, Nyström interpolants based on the zeros of Legendre polynomials for a non-compact integral operator equation, *IMA J. Numer. Anal.* 14 (1993) 81–95.
- [15] G. Monegato, A stable Nyström interpolant for some Mellin convolution equations, *Numer. Algorithms* 11 (1996) 271–283.
- [16] G. Monegato, S. Prössdorf, On the numerical treatment of an integral equation arising from a cruciform crack problem, *Math. Methods Appl. Sci.* 12 (1990) 489–502.
- [17] M.P. Stallybrass, A pressurized crack in the form of a cross, *Quart. J. Mech. Appl. Math.* 23 (1970) 35–48.