On the simultaneous approximation of a Hilbert transform and its derivatives on the real semiaxis☆

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Abstract

In this paper we propose a global method to approximate the derivatives of the weighted Hilbert transform of a given function f

$$\mathbf{H}_p(fw_\alpha,t) \quad = \quad \frac{d^p}{dt^p}\int_0^{+\infty}\frac{f(x)}{x-t}w_\alpha(x)dx = p!\,\mathop{=\!\!\!\!\!\!\int}_0^{+\infty}\frac{f(x)}{(x-t)^{p+1}}w_\alpha(x)dx,$$

where $p\in\{1,2,\dots\},\ t>0$, and $w_\alpha(x)=e^{-x}x^\alpha$ is a Laguerre weight. The right-hand integral is defined as the finite part in the Hadamard sense. The proposed numerical approach is convenient when the approximation of the function $\mathbf{H}_p(fw_\alpha,t)$ is required. Moreover, if there is the need, all the computations can be performed without differentiating the density function f. Numerical stability and convergence are proved in suitable weighted uniform spaces and numerical tests which confirm the theoretical estimates are presented.

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1. Introduction

The paper is devoted to the approximation of the derivatives of the weighted Hilbert transform of f

$$\mathbf{H}_p(fw,t) \quad = \quad \frac{d^p}{dt^p}\int_0^{+\infty}\frac{f(x)}{x-t}w(x)dx = p!\,\mathop{=\!\!\!\!\!\!\int}_0^{+\infty}\frac{f(x)}{(x-t)^{p+1}}w(x)dx, \tag{1}$$

where $p\in\{1,2,\dots\},\ t>0,\ w(x):=w_\alpha(x)=e^{-x}x^\alpha$ is a Laguerre weight. The integral in (1) can be also defined as a finite part integral in the Hadamard sense (see [7],[16]). Integrals of the type (1) appear for instance in hypersingular integral equations, models for many problems in Physics and Engineering areas (see [16] and the reference therein, [5], [10], [1]). Usually, in the literature, quadrature rules are proposed for the approximation of $\mathbf{H}_p(fw,t)$ for any fixed t. Instead, in the present paper, setting

$$\begin{aligned}\mathbf{H}_p(fw,t) \quad &= \quad \frac{d^p}{dt^p}\left(\int_0^{+\infty}\frac{f(x)-f(t)}{x-t}w(x)dx+f(t)\int_0^{+\infty}\frac{w(x)}{x-t}dx\right)\\ &=: \quad \frac{d^p}{dt^p}\mathbf{F}(f,t)+\frac{d^p}{dt^p}\left(f(t)\mathbf{H}_0(w,t)\right),\end{aligned} \tag{2}$$

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we propose to approximate the function $\mathbf{F}^{(p)}(f)$ by the $p-$th derivative of a suitable Lagrange polynomial interpolating $\mathbf{F}(f)$ at Laguerre zeros. For a correct error estimate in weighted uniform spaces, at first we determine the class of $\mathbf{F}(f)$ depending on the Zygmund-type space f belongs to. Since in the general case the samples of $\mathbf{F}(f)$ at the interpolation knots cannot be exactly computed, we approximate them by a truncated Gauss-Laguerre rule (see [12]). Moreover, by reusing the same interpolation knots, it is possible approximate also the $p-$th derivative of the function $f(t)\mathbf{H}_0(w,t)$, avoiding the differentiation of the density function f.

This procedure is especially advisable when the approximation of $\mathbf{H}_p(fw,t)$ is required for a "large" number of t and/or the uniform convergence of the rule to $H_p(fw)$ is needed. This happens, for instance, when (1) appears in a hypersingular integral equation and in order to solve it one wants to use a collocation method.

The paper is organized as follows. In Section 2 are collected some auxiliary results and notations. Section 3 provides the exposition of the numerical methods and results about the stability and the convergence, with error estimates in some weighted uniform spaces. Section 4 contains a brief description of computational details in the implementation process. In Section 5 some numerical experiments are discussed and comparisons with some standard numerical methods are shown. Finally in Section 6 the proofs of our main results are stated.

2. Basic results and properties

Along all the paper the constant $\mathcal{C}$ will be used several times, having different meaning in different formulas. Moreover from now on we will write $\mathcal{C} \neq \mathcal{C}(a,b,\ldots)$ in order to say that $\mathcal{C}$ is a positive constant independent of the parameters $a,b,\ldots$, and $\mathcal{C} = \mathcal{C}(a,b,\ldots)$ to say that $\mathcal{C}$ depends on $a,b,\ldots$. Moreover, if $A,B \geq 0$ are quantities depending on some parameters, we will write $A \sim B$, if there exists a constant $0 < \mathcal{C} \neq \mathcal{C}(A,B)$ such that $\frac{B}{\mathcal{C}} \leq A \leq \mathcal{C}B$. Finally, $\mathbb{P}_m$ will denote the space of the algebraic polynomials of degree at most m.

Let $w(x) = e^{-x}x^{\alpha}$ be the Laguerre weight of parameter $\alpha > -1$ and let $\{p_m(w)\}_m$ be the corresponding sequence of orthonormal polynomials with positive leading coefficients. Let us denote by $\{x_{m,k}\}_{k=1}^{m}$ the zeros of $p_m(w)$ in increasing order, i.e. $x_{m,k} < x_{m,k+1}, k = 1,\ldots,m-1$. From now on, for any fixed $0 < \theta < 1$, the integer j will denote the index of the zero of $p_m(w)$ s. t.

$$j := j(m) = \min_{k=1,2,..,m} \{k : x_{m,k} \geq 4m\theta\}. \tag{3}$$

With $u(x) = x^{\gamma}e^{-x/2}, \gamma \geq 0$, we will consider

$$C_u = \begin{cases} \{f \in C^0((0,\infty)) : \lim\limits_{\substack{x\to+\infty \\ x\to 0^+}} (fu)(x) = 0\}, & \gamma > 0, \\ \left\{f \in C^0([0,\infty)) : \lim\limits_{x\to+\infty} (fu)(x) = 0\right\}, & \gamma = 0, \end{cases}$$

equipped with the norm

$$\|f\|_{C_u} = \|fu\| := \|fu\|_{\infty} = \sup_{x\geq 0} |(fu)(x)|,$$

where $C^0(E)$ is the space of the continuous functions on the set E. Sometimes, for the sake of brevity, we will use $\|f\|_E = \sup_{x\in E}|f(x)|$.

For smoother functions, we introduce the Sobolev-type spaces of order $r \in \mathbb{N}$

$$W_r(u) = \left\{f \in C_u \ : \ f^{(r-1)} \in AC(0,+\infty) \text{ and } \|f^{(r)}\varphi^r u\| < +\infty\right\},$$

where $\varphi(x) = \sqrt{x}$ and $AC((0,+\infty))$ is the set of the absolutely continuous functions on every closed subset of $(0,+\infty)$. We equip them with the norm

$$\|f\|_{W_r(u)} := \|fu\| + \|f^{(r)}\varphi^r u\|.$$

In what follows $W_0(u) = C_u$. For any $f \in C_u$ and for any $t > 0$, let

$$\Omega_\varphi^r(f,t)_u = \sup_{0<h\le t} \|u\Delta_{h\varphi}^r f\|_{I_{rh}}$$

be the main part of the $r-$th $\varphi-$modulus of smoothness, where $I_{rh} = \left[4r^2h^2, \frac{\mathcal{C}}{h^2}\right]$, being $\mathcal{C}$ a fixed positive constant, and

$$\Delta_{h\varphi}^r f(x) = \sum_{k=0}^{r} (-1)^k \binom{r}{k} f\left(x + h\varphi(x)(r-k)\right).$$

By means of $\Omega_\varphi^r(f,t)_u$ we define the Zygmund-type spaces

$$Z_\lambda(u) = \left\{ f \in C_u \ : \ \sup_{t>0} \frac{\Omega_\varphi^r(f,t)_u}{t^\lambda} < +\infty \right\}$$

of parameter $0 < \lambda < r$, equipped with the norm

$$\|f\|_{Z_\lambda(u)} = \|fu\| + \sup_{t>0} \frac{\Omega_\varphi^r(f,t)_u}{t^\lambda}.$$

Now we give a result which can be useful in several contexts

Lemma 2.1. *Let $u(x) = e^{-\frac{x}{2}}x^\gamma$, $0 \le \gamma \le 1$. Then, for $0 < \lambda < 1$ and $p \in \mathbb{N}$, $f^{(p)} \in Z_\lambda(u\varphi^p)$ implies $f \in Z_{\lambda+p}(u)$ and viceversa.*

Denoting by $E_m(f)_u = \inf_{P\in\mathbb{P}_m} \|(f-P)u\|$, the error of best polynomial approximation in C_u, for any $f \in W_r(u)$ the following estimates hold [3]

$$E_m(f)_u \le \frac{\mathcal{C}}{(\sqrt{m})^r} E_{m-r}(f^{(r)})_{u\varphi^r}, \tag{4}$$

$$E_m(f)_u \le \mathcal{C}\frac{\|f^{(r)}u\varphi^r\|}{(\sqrt{m})^r}, \tag{5}$$

where $\mathcal{C} \neq \mathcal{C}(m,f)$.

Let $\mathcal{L}_{m+1}(w,g)$ be the Lagrange polynomial interpolating a given function g at the zeros of $p_m(w,x)(4m-x)$. Let j be defined in (3) and let χ_j be the characteristic function of the segment $(0, x_{m,j})$. The Lagrange polynomial $L_{m+1}(w,g) := \mathcal{L}_{m+1}(w, g\chi_j)$ introduced in [9] (see also [11]) can be expressed as

$$L_{m+1}(w,g,x) = \sum_{k=1}^{j} l_{m,k}(x)\frac{4m-x}{4m-x_{m,k}} f(x_{m,k}) =: \sum_{k=1}^{j} \ell_{m,k}(x) f(x_{m,k}), \tag{6}$$

where $l_{m,k}(x) = \frac{p_m(w,x)}{p_m'(x_{m,k})(x-x_{m,k})}$. $L_{m+1}(w,g)$ belongs to $\mathcal{P}_m^* \subset \mathbb{P}_m$, with

$$\mathcal{P}_m^* = \{p \in \mathbb{P}_m : p(x_{m,k}) = p(4m) = 0, \quad k > j\},$$

and the operator $L_{m+1}(w)$ projects C_u onto $\mathcal{P}_m^*$.

About the simultaneous approximation we state the following result

Theorem 2.2. *If $f \in Z_{p+\lambda}(u)$ with $0 < \lambda < 1$, $k, p \in \mathbb{N}, k \le p$ and α, γ are two real parameters satisfying the inequality*

$$\frac{\alpha}{2} + \frac{1}{4} \le \gamma \le \frac{\alpha}{2} + \frac{5}{4}, \tag{7}$$

we have

$$\|(f - L_{m+1}(w,f))^{(k)}\varphi^k u\| \le \mathcal{C}\left\{ \frac{\log m}{(\sqrt{m})^{p+\lambda-k}} \|f\|_{Z_{p+\lambda}(u)} + e^{-Am}\|fu\| \right\},$$

where $0 < \mathcal{C} \neq \mathcal{C}(m, f)$. *In particular, if* $f \in W_{p+r}(u)$, $r \geq 1$,

$$\|(f - L_{m+1}(w, f))^{(k)} \varphi^k u\| \leq \mathcal{C} \frac{\log m}{(\sqrt{m})^{p+r-k}} \left\{ \|f\|_{W_{p+r}(u)} + e^{-Am} \|fu\| \right\}, \quad 0 < \mathcal{C} \neq \mathcal{C}(m, f). \tag{8}$$

Nevertheless, if the parameters α and γ do not satisfy the assumptions of the previous theorem, it is possible to modify the interpolation process making use of the *method of additional knots*. To our aims, here we consider the case with one additional knot. Setting $x_{m,0} = \frac{x_{m,1}}{2}$, let $L_{m+1,1}(w, f)$ be the truncated Lagrange polynomial interpolating f at the zeros of $Q_{m+2}(x) := \left(x - \frac{x_{m,1}}{2}\right) p_m(w, x)(4m - x)$ (see [18], [13]), i.e.

$$\begin{aligned} L_{m+1,1}(w, f, x) &= \sum_{k=0}^{j} \tilde{\ell}_{m,k}^{(p)}(x) f(x_{m,k}), \qquad (9) \\ & \qquad \tilde{\ell}_{m,k}(x) = \frac{Q_{m+2}(x)}{Q'_{m+2}(x_{m,k})(x - x_{m,k})}, \quad k = 0, 1, \ldots, j. \end{aligned}$$

Also in this case we state a result dealing with the simultaneous approximation

Theorem 2.3. *If* $f \in W_{p+r}(u)$, *with* $r \geq 1$, $k, p \in \mathbb{N}, k \leq p$ *and* α, γ *are two real parameters satisfying the inequality*

$$\frac{\alpha}{2} - \frac{3}{4} \leq \gamma \leq \frac{\alpha}{2} + \frac{1}{4},$$

we have

$$\|(f - L_{m+1,1}(w, f))^{(k)} \varphi^k u\| \leq \mathcal{C} \left\{ \frac{\log m}{(\sqrt{m})^{p+r-k}} \|f\|_{W_{p+r}(u)} + e^{-Am} \|fu\| \right\},$$

where $0 < \mathcal{C} \neq \mathcal{C}(m, f)$.

Denoting by $\{\lambda_{m,k}\}_{k=1}^m$ the Christoffel numbers w.r.t. w, we recall the "Truncated Gauss-Laguerre rule" [12] based on the first j zeros of $p_m(w)$, with j defined in (3),

$$\int_0^{+\infty} f(x) w(x) dx = \sum_{k=1}^{j} f(x_{m,k}) \lambda_{m,k} + R_m(f). \tag{10}$$

3. The Method

At first we describe the method assuming that the functions $\{f^{(k)}(t)\}_{k=1}^p$ are explicitly known.

Let $L_{m+1}(w, \mathbf{F}(f))$ be the truncated Lagrange polynomials interpolating $\mathbf{F}(f)$ at $\{x_{m,k}\}_{k=1}^m \cup \{4m\}$. In (2) we approximate $\mathbf{F}^{(p)}(f, t)$ with $L_{m+1}^{(p)}(w, \mathbf{F}(f), t)$. To compute the quantities $\mathbf{F}(f, x_{m,k}), k = 1, \ldots, j$, with j defined in (3), we will use the rule (10) based on the first q zeros of $p_{m+1}(w)$, where for any fixed $0 < \theta < 1$

$$x_{m+1,q} = \min \left\{ x_{m+1,k} : x_{m+1,k} \geq \theta 4(m+1), \quad k = 1, 2, .., m+1 \right\},$$

and we replace $\mathbf{F}(f, x_{m,k})$ with

$$\mathbf{F}_m(f, x_{m,k}) := \sum_{i=1}^{q} \lambda_{m+1,i} \frac{f(x_{m+1,i}) - f(x_{m,k})}{x_{m+1,i} - x_{m,k}}, \quad k = 1, \ldots, j.$$

Therefore we have

$$\mathbf{H}_p(fw, t) = \Phi_{p,m}(f, t) + \sum_{k=0}^{p} \binom{p}{k} f^{(k)}(t) \mathbf{H}_{p-k}(w, t) + \rho_{p,m}(f, t) =: \mathbf{H}_{p,m}(f, t) + \rho_{p,m}(f, t), \tag{11}$$

where

$$\Phi_{p,m}(f,t)=\sum_{k=1}^{j}\ell_{m,k}^{(p)}(t)\sum_{i=1}^{q}\lambda_{m+1,i}\frac{f(x_{m+1,i})-f(x_{m,k})}{x_{m+1,i}-x_{m,k}}.$$

Firstly we observe that in view of [2, Lemma 2.1], Gaussian knots and interpolation nodes are sufficiently far among them. This good "distance" prevents numerical cancellation when computing $f(x_{m+1,i})-f(x_{m,k})$ and $x_{m+1,i}-x_{m,k}$. About the stability and the convergence of the procedure, we are able to prove the following results

Theorem 3.1. *Let $p\in I\!N_0$ and $\alpha\geq\frac{1}{2}$. Then, for any function $f\in Z_{p+\lambda}(u)$ with $0<\lambda<1$, if the parameters α,γ satisfy the assumption*

$$\frac{\alpha}{2}+\frac{1}{4}\leq\gamma\leq\min\left(\alpha,\frac{\alpha}{2}+\frac{5}{4}\right), \tag{12}$$

we have

$$\|\Phi_{p,m}(f)u\varphi^p\|\leq\mathcal{C}\|f\|_{Z_{p+\lambda}(u)}\log^2 m,\quad 0<\mathcal{C}\neq\mathcal{C}(m,f). \tag{13}$$

Moreover, if $f\in Z_{p+r+\lambda}(u), r\geq 1$, then

$$\|\rho_{p,m}(f)u\varphi^p\|\leq\mathcal{C}\|f\|_{Z_{p+r+\lambda}(u)}\frac{\log^2 m}{(\sqrt{m})^r},\quad 0<\mathcal{C}\neq\mathcal{C}(m,f). \tag{14}$$

Corollary 3.1. *Let $p\in I\!N_0$. Then, under the assumption* (12) *we have*

$$\|\Phi_{p,m}(f)u\varphi^p\|\leq\mathcal{C}\|f\|_{W_{p+1}(u)}\log^2 m,\quad\forall f\in W_{p+1}(u),\quad 0<\mathcal{C}\neq\mathcal{C}(m,f),$$

and if $f\in W_{p+r+1}(u), r\geq 1$, then

$$\|\rho_{p,m}(f)u\varphi^p\|\leq\mathcal{C}\|f\|_{W_{p+r+1}(u)}\frac{\log^2 m}{(\sqrt{m})^r},\quad 0<\mathcal{C}\neq\mathcal{C}(m,f).$$

In order to treat the case $\alpha<\frac{1}{2}$, we slightly modify the interpolation process making use of the method of additional nodes. By arguments similar to those used in the previous case, we approximate $\mathbf{F}^{(p)}(f,t)$ by $L_{m+1,1}^{(p)}(w,\mathbf{F}(f),t)$, where $L_{m+1,1}(w,\mathbf{F}(f))$ is defined in (9). The samples $\mathbf{F}(f,x_{m,k}), k=0,\ldots,j$, are approximated by the truncated Gaussian rule (10). Thus we have

$$\mathbf{H}_p(fw,t)\quad=\quad\Phi_{p,m}^{(1)}(f,t)+\sum_{k=0}^{p}\binom{p}{k}f^{(k)}(t)\mathbf{H}_{p-k}(w,t)+\rho_{p,m}^{(1)}(f,t)=:\mathbf{H}_{p,m}^{(1)}(f,t)+\rho_{p,m}^{(1)}(f,t), \tag{15}$$

where

$$\Phi_{p,m}^{(1)}(f,t)\quad=\quad\sum_{k=0}^{j}\tilde{\ell}_{m,k}^{(p)}(t)\sum_{i=1}^{q}\lambda_{m+1,i}\frac{f(x_{m+1,i})-f(x_{m,k})}{x_{m+1,i}-x_{m,k}}.$$

In this case we can prove the following result

Theorem 3.2. *Let $p\in I\!N_0$ and $-\frac{1}{4}<\alpha<\frac{1}{2}$. Then, for any function $f\in W_{p+1}(u)$, under the assumption*

$$0\leq\gamma<\min\left\{\alpha,\frac{\alpha}{2}\right\}+\frac{1}{4},$$

we have

$$\|\Phi_{p,m}(f)^{(1)}u\varphi^p\|\leq\mathcal{C}\mathcal{M}\log m, \tag{16}$$

where $\mathcal{C}\neq\mathcal{C}(m,f)$ and $\mathcal{M}=\max\left(\|\mathbf{F}_m(f)\|_{W_p(u)},\|f\|_{W_{p+r+1}(u)}\right)$.

Moreover, if $f \in W_{p+r+1}(u), r \geq 1,$ *then*

$$\|\rho_{p,m}^{(1)}(f)u\varphi^p\| \leq \mathcal{C}\frac{\log m}{(\sqrt{m})^r}\|f\|_{W_{p+r+1}(u)}, \tag{17}$$

where $0 < \mathcal{C} \neq \mathcal{C}(m, f)$.

Now we discuss how to proceed when the derivatives $\{f^{(k)}(t)\}_{k=1}^p$ are not available. In this case, we propose to approximate $f^{(k)}(t)$ by $L_{m+1}^{(k)}(w, f, t)$ for $k = 1, 2, \ldots, p$, where $L_{m+1}(w, f)$ has been defined is the truncated Lagrange polynomial defined in (6). By this way, reusing the same samples involved in the evaluation of $\Phi_{p,m}(f, t)$ and taking into account that $\mathbf{H}_{p-k}(w, t)$ can be computed with the desired accuracy, we get

$$\mathbf{H}_p(fw, t) \quad = \quad \Phi_{p,m}(f, t) + f(t)\mathbf{H}_p(w, t) + \sum_{k=1}^{p}\binom{p}{k}L_{m+1}^{(k)}(w, f, t)\mathbf{H}_{p-k}(w, t) + \Psi_{p,m}(f, t), \tag{18}$$

where

$$\Psi_{p,m}(f, t) \quad = \quad \rho_{p,m}(f, t) + \sum_{k=1}^{p}\binom{p}{k}\tau_{k,m}(f, t)\mathbf{H}_{p-k}(w, t),$$

with

$$\tau_{k,m}(f, t) = (f(t) - L_{m+1}(w, f, t))^{(k)}.$$

The next Theorem deals with the pointwise estimate of the error $\Psi_{p,m}(f, t)$:

Theorem 3.3. *Let* $p \geq 1$. *Then, for any function* $f \in Z_{p+r+\lambda}(u)$ *with* $0 < \lambda < 1$, $r \geq 1$, *under the assumption* (12) *we have*

$$|\Psi_{p,m}(f, t)u(t)\varphi^p(t)| \leq \mathcal{C}\frac{\|f\|_{Z_{p+r+\lambda}(u)}}{(\sqrt{m})^r}\log^2 m\ \ (1 + \Theta(t)), \quad 0 < \mathcal{C} \neq \mathcal{C}(m, f),$$

where

$$\Theta(t) = \begin{cases} t^{-\frac{p-1}{2}}, & 0 < t < 1 \\ 1, & t \geq 1 \end{cases}.$$

Remark 3.1. *In any closed subset* $[a, b] \subset \left(\frac{c}{m}, +\infty\right)$, *c being an arbitrary positive constant, we get*

$$|\Psi_{p,m}(f, t)u(t)\varphi^p(t)| \leq \mathcal{C}\frac{\|f\|_{Z_{p+r+\lambda}(u)}}{(\sqrt{m})^r}\log^2 m, \quad 0 < \mathcal{C} \neq \mathcal{C}(m, f).$$

Remark 3.2. *We point out that there exist other sets of interpolation nodes which have optimal Lebesgue constants and that are sufficiently far among from the Gaussian knots. For instance, the zeros of* $p_m(\bar{w})$ *with* $\bar{w}(x) = xw(x)$ *(see [17]).*

4. Computational details

About the computation of the derivatives of the fundamental Lagrange polynomials $\{\ell_{m,k}^{(i)}(t)\}_{k=1}^j$, $i = 1, 2, \ldots, p$, it is no hard to prove that

$$\ell_{m,k}^{(i)}(t) = \frac{(4m - x)}{(4m - x_{m,k})}l_{m,k}^{(i)}(t) - \frac{i}{(4m - x_{m,k})}l_{m,k}^{(i-1)}(t)$$

and setting $(j)_i := j(j-1)\dots(j-i+1)$,

$$l_{m,k}^{(i)}(t) = \lambda_{m,k} \sum_{j=i}^{m-1} \sqrt{(j)_i} p_j(w, x_{m,k}) p_{j-i}(w_{\alpha+i}, t), \quad i = 1, 2, \dots, p.$$

In order to compute the Hadamard transforms of the weight $w(x) = e^{-x}x^\alpha$, $\alpha > -1$, taking into account that

$$\int\!\!\!\!\!\!= _0^{+\infty} \frac{g(x)}{(x-t)^{p+1}} dx := \frac{1}{p!} \frac{d^p}{dt^p} \int\!\!\!\!\!\!- _0^{+\infty} \frac{g(x)}{x-t} dx,$$

we use the expressions of $\mathbf{H}_0(w,t)$ in [18] and [6, p. 1086, 9.213]

$$\frac{d}{dt} Ei(t) = -\frac{d}{dt} \int_{-t}^{+\infty} \frac{e^{-x}}{x} dx = \frac{e^t}{t}, \qquad \frac{d}{dt} {}_1F_1(a,b;t) = \frac{a}{b} {}_1F_1(a+1, b+1, t).$$

We performed the computation of the Exponential integral function Ei and of the Kummer Confluent Hypergeometric function ${}_1F_1$ by the *Wolfram Mathematica* routines `ExpIntegralEi` and `Hypergeometric1F1`, respectively.

5. Numerical tests

In this section we show the effectiveness of the rules $\mathbf{H}_{p,m}(fw)$ and $\mathbf{H}_{p,m}^{(1)}(fw)$ in (11) and (15), respectively, according to the value of α, by producing some numerical tests. Since the exact values of the integrals are unknown, we will retain as exact those values computed with $m = 1000$ and we will set

$$\bar{\rho}_{p,m}(f,t) = |\Phi_{p,m}(f,t) - \Phi_{p,1000}(f,t)|,$$

$$\bar{\rho}_{p,m}^{(1)}(f,t) = |\Phi_{p,m}^{(1)}(f,t) - \Phi_{p,1000}(1)(f,t)|.$$

We remark that, in each of the examples, the "truncation intervals", which depend on θ, have been empirically detected. More precisely, in each table the indices j and q have been chosen as

$$j := \max_{k=1,\dots,m} \left\{ k \ : \ \left| \ell_{m+1,k}^{(p)}(t) \sum_{i=1}^{q} \lambda_{m+1,i} \frac{f(x_{m+1,i}) - f(x_{m,k})}{x_{m+1,i} - x_{m,k}} \right| \geq eps \right\},$$

and

$$q := \max_{k=1,\dots,m} q_k,$$

with

$$q_k := \max_{i=1,\dots,m} \left\{ i \ : \ \left| \lambda_{m+1,i} \frac{f(x_{m+1,i}) - f(x_{m,k})}{x_{m+1,i} - x_{m,k}} \right| \geq eps \right\},$$

respectively. An analogous computation of the index j holds when we use $\Phi_{p,m}^{(1)}(f,t)$.

In the first example we also compare the results obtained by (11) with those achieved properly adapting some of the numerical methods collected in [15] for finite part integrals on bounded intervals. More precisely, by an idea in [8] (see also [15]), starting from the decomposition

$$\begin{aligned} \mathbf{H}_p(fw,t) &= \int_0^{+\infty} \frac{f(x) - \sum_{k=0}^{p} \frac{f^{(k)}(t)}{k!}(x-t)^k}{(x-t)^{p+1}} w(x) dx + \sum_{k=0}^{p} \frac{f^{(k)}(t)}{k!} \int\!\!\!\!\!\!= _0^{+\infty} \frac{w(x)}{(x-t)^{p+1-k}} dx \\ &=: \mathbf{F}_p(f,t) + \sum_{k=0}^{p} \frac{f^{(k)}(t)}{k!} \mathbf{H}_{p-k}(w,t), \end{aligned}$$

we approximate $\mathbf{F}_p(f,t)$ by

$$Q_{p,m}(f,t) := \sum_{i=1}^{q^*} \frac{f(x_{m,i}) - \sum_{k=0}^{p} \frac{f^{(k)}(t)}{k!}(x_{m,i}-t)^k}{(x_{m,i}-t)^{p+1}} \lambda_{m,i},$$

with

$$q^* := \max_{i=1,\dots,m} \left\{ i \ : \ \left| \frac{f(x_{m,i}) - \sum_{k=0}^{p} \frac{f^{(k)}(t)}{k!}(x_{m,i}-t)^k}{(x_{m,i}-t)^{p+1}} \lambda_{m,i} \right| \geq eps \right\}.$$

Unfortunately the above rule suffers from numerical instability when t is very close to one of the quadrature knots $x_{m,i}, i = 1, \dots, j$. In order to highlight this shortcoming, in Figure 1 we show the absolute errors $|Q_{p,m}(f,t) - Q_{p,1000}(f,t)|$ obtained in the first example for increasing values of m at the points $t = 2.007880721659913$ (right) and $t = 4.4971303840560 21e-001$ (left). As one can see there are some picks due to the numerical cancellation phenomenon.

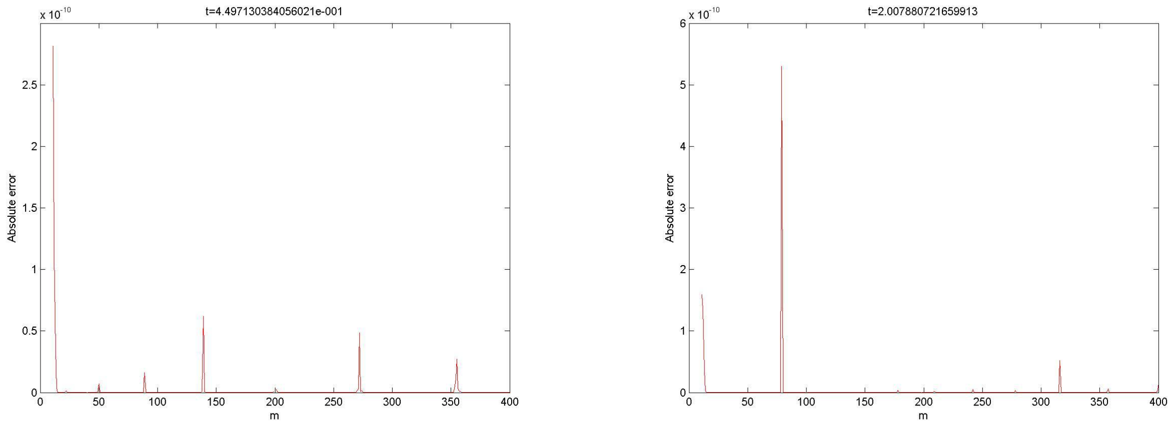


Figure 1: Absolute errors by the rule $Q_{p,m}(f,t)$, for increasing values of m

In order to overcome this numerical instability, following an idea in [15], we break up the interval $(0,+\infty)$ into $(0,t)$ and $(t,+\infty)$. By making the changes of variable $x = \frac{t}{2}(y+1) =: \psi_1(y)$ and $x = y + t =: \psi_2(y)$ in the intervals $(0,t)$ and $(t,+\infty)$, respectively, we get

$$\begin{aligned}
\mathbf{H}_p(fw,t) &= \left(\frac{2}{t}\right)^{p-\alpha} \int_{-1}^{1} \frac{f(\psi_1(y)) - \sum_{k=0}^{p} \frac{f^{(k)}(t)}{k!}(\psi_1(y)-t)^k}{(y-1)^{p+1}} e^{-\psi_1(y)}(1+y)^\alpha dy \\
&+ e^{-t} \int_0^{+\infty} \frac{f(\psi_2(y)) - \sum_{k=0}^{p} \frac{f^{(k)}(t)}{k!} y^k}{y^{p+1}} \psi_2(y)^\alpha e^{-y} dy + \sum_{k=0}^{p} \frac{f^{(k)}(t)}{k!} \mathbf{H}_{p-k}(w,t) \\
&=: G_1(f,t) + G_2(f,t) + \sum_{k=0}^{p} \frac{f^{(k)}(t)}{k!} \mathbf{H}_{p-k}(w,t).
\end{aligned}$$

Then, we use the m-th Gauss-Jacobi rule w. r. t. the weight $(1+y)^\alpha$ to approximate $G_1(f,t)$ and the m-th Truncated Gauss-Laguerre rule w. r. t. the weight e^{-y} on the first j^* Gaussian nodes to approximate $G_2(f,t)$. Thus we obtain

$$\begin{aligned}
\mathbf{H}_p(fw,t) &= G_{p,m}(f,t) + \sum_{k=0}^{p} \frac{f^{(k)}(t)}{k!} \mathbf{H}_{p-k}(w,t) + e_{p,m}(f,t) \\
&:= \bar{\mathbf{H}}_{p,m}(f,t) + e_{p,m}(f,t). \qquad (19)
\end{aligned}$$

By inspecting Tables 1-2, it seems that our rule in (11) is faster than the rule (19) for t "close" to 0. However theoretical estimates which confirm this worse behavior are not available. In addition, the computation of $G_{p,m}(f,t)$ requires many more evaluations of f than those employed in $\Phi_{p,m}(f,t)$. Indeed, for s values of t the computation of $G_{p,m}(f,t)$ requires $(m+j^*)s$ samples of f, while that of $\Phi_{p,m}(f,t)$ can be performed with only $n=(j+q)$ evaluations of f, with n independent of s.

We will set

$$\bar{e}_{p,m}(f,t)=|\bar{\mathbf{H}}_{p,m}(f,t)-\bar{\mathbf{H}}_{p,1000}(f,t)|.$$

All the computations have been performed in double-machine precision ($eps \approx 2.22044e-16$) and in all the tables the symbol "-" means that the machine accuracy has been achieved.

Finally, since the results obtained by formulae (11) and (18) are comparable, we will report in the next tables the errors obtained by (11).

Example 5.1.

$$H_p(fw,t)=\int_0^{+\infty}\frac{\sin(x+5)}{(x-t)^{p+1}}x^{0.6}e^{-x}dx, \quad \alpha=0.6, \quad p=0,1,2.$$

Since the function $f(x)=\sin(x+5)$ is very smooth we expect for a fast convergence. Indeed, looking at Table 1, in different values of t, we get approximations of the integrals with the machine precision taking $m=110$ but only $j=62$ and $q=47$.

m	j	q	$\bar{\rho}_{0,m}(f,0.01)$	$\bar{\rho}_{0,m}(f,0.1)$	$\bar{\rho}_{0,m}(f,1)$	$\bar{\rho}_{0,m}(f,5)$
20	20	19	$1.8707e-4$	2.8917 e - 5	$1.3495e-5$	$23728e-5$
40	35	28	$1.4234e-7$	3.0965 e - 8	$8.23754e-10$	$4.5692e-10$
80	53	40	$8.1953e-14$	1.6356 e- 14	$9.01422e-15$	$2.30113e-14$
100	59	45	$-$	$-$	$-$	$-$
m	j	q	$\bar{\rho}_{1,m}(f,0.01)$	$\bar{\rho}_{1,m}(f,0.1)$	$\bar{\rho}_{1,m}(f,1)$	$\bar{\rho}_{1,m}(f,5)$
20	20	19	$2.4888e-3$	1.1380 e - 3	$1.2213e-4$	$9.6884e-6$
40	35	28	$3.9794e-6$	5.0724 e - 7	$1.2217e-7$	$1.2751e-7$
80	53	40	$5.2234e-12$	5.8445 e - 13	$4.8632e-14$	$6.4330e-14$
100	59	45	$5.4358e-15$	$-$	$-$	$-$
m	j	q	$\bar{\rho}_{2,m}(f,0.01)$	$\bar{\rho}_{2,m}(f,0.1)$	$\bar{\rho}_{2,m}(f,1)$	$\bar{\rho}_{2,m}(f,5)$
20	20	19	$9.4949e-3$	5.7418e -3	$8.2368e-5$	$6.0743e-5$
40	35	28	$3.1383e-5$	1.0027e - 5	$5.8862e-8$	$3.3773e-8$
80	53	40	$8.7427e-11$	2.1514 e -12	$3.4513e-13$	$2.0412e-13$
100	59	45	$1.1752e-13$	3.8601 e - 15	$-$	$-$
110	62	47	$-$	$-$	$-$	$-$

Table 1: Example 5.1: $\bar{\rho}_{p,m}(f,t),\ p=0,1,2,$ with $t=0.01,0.1,1,5$

As announced, in Table 2, we report the errors $\bar{e}_{p,m}(f,t),\ p=0,1,2,$ obtained using the rule $\bar{\mathbf{H}}_{p,m}(f,t)$ in (19). As one can see, for values of t close to 0, the convergence order seems to be lower than the one of our rule (11).

In Figure 2, we show the graphs of the functions $\mathbf{H}_{p,110}(f,t), p=0,1,2.$

Example 5.2.

$$H_p(fw,t)=\int_0^{+\infty}\frac{e^{-\frac{x}{2}}x^{\frac{5}{4}}}{(x-t)^{p+1}(1+x^2)^4}dx, \quad \alpha=\frac{5}{4}, \quad p=0,1,2.$$

In this case $f(x)=\frac{e^{\frac{x}{2}}}{(1+x^2)^4}\in W_{13}(u)$, with $\gamma=\frac{7}{8}$, and, according to (14), the error behaves like $\|f\|_{W_{13}(u)}\frac{\log^2 m}{(\sqrt{m})^{12-p}}$ with $\|f\|_{W_{13}(u)}\sim 10^9$. This convergence behavior is confirmed by the numerical results presented in Table 3, in fact it is necessary to increase m in order to attain exact decimal digits and with $m=900$ but only $j=392$ and $q=280$ we get approximations with almost the machine precision.

The graphs of the functions $H_{p,900}(f,t), p=0,1,2,$ are shown in Figure 3.

m	j	$\bar{e}_{0,m}(f,0.01)$	$\bar{e}_{0,m}(f,0.1)$	$\bar{e}_{0,m}(f,1)$	$\bar{e}_{0,m}(f,5)$
10	10	$4.1098e-4$	9.3945 e - 5	$1.8534e-7$	$1.2374e-9$
40	26	$2.7996e-5$	5.6939 e - 7	$1.8750e-13$	$-$
80	37	$4.9615e-6$	1.1648 e - 8	$-$	$-$
m	j	$\bar{e}_{1,m}(f,0.01)$	$\bar{e}_{1,m}(f,0.1)$	$\bar{e}_{1,m}(f,1)$	$\bar{e}_{1,m}(f,5)$
10	10	$8.3988e-4$	1.7563 e - 4	$9.5187e-8$	$2.9087e-11$
40	26	$4.9043e-5$	8.8342 e - 7	$9.0383e-14$	$-$
80	37	$8.4866e-6$	1.7470 e -8	$-$	$-$
m	j	$\bar{e}_{2,m}(f,0.01)$	$\bar{e}_{2,m}(f,0.1)$	$\bar{e}_{2,m}(f,1)$	$\bar{e}_{2,m}(f,5)$
10	10	$7.6818e-5$	1.9502 e - 5	$3.7848e-8$	$2.0333e-12$
40	26	$4.8308e-6$	1.0652 e -7	$4.1630e-14$	$-$
80	37	$8.4616e-7$	2.1410 e -9	$-$	$-$

Table 2: Example 5.1: $\bar{e}_{p,m}(f,t),\ p=0,1,2$, with $t=0.01,0.1,1,5$

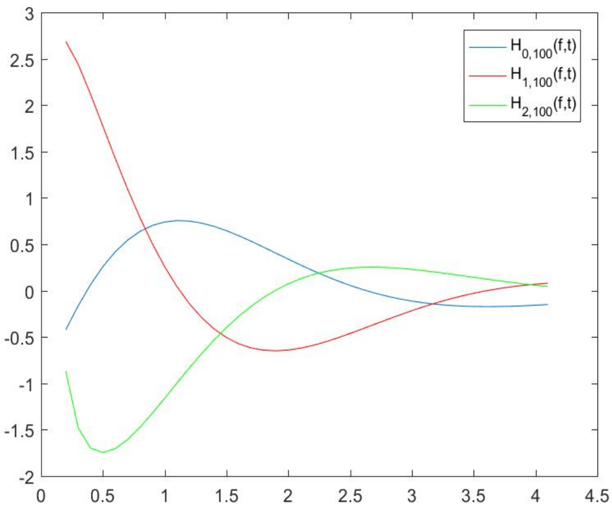


Figure 2: Example 1

Example 5.3. Consider the integrals

$$H_p(fw_{\frac{1}{2}},t)=\int_0^{+\infty}\frac{\sin\left(\frac{x}{4}\right)\left|x-\frac{1}{2}\right|^{\frac{11}{2}}}{(x-t)^{p+1}}x^{\frac{5}{2}}e^{-x}dx,\quad \alpha=\frac{5}{2},\quad p=0,1.$$

Here $f(x)=\sin\left(\frac{x}{4}\right)\left|x-\frac{1}{2}\right|^{\frac{11}{2}}\in Z_{5+\frac{1}{2}}(u)$, with $\gamma=\frac{3}{2}$, and the numerical error, as Table 4 shows, agrees with the theoretical estimate $\frac{\log^2 m}{(\sqrt{m})^{5-p}}$.

In Figure 4, we show the graphs of the functions $H_{p,900}(f,t), p=0,1$.

Example 5.4. As last example we consider the integrals

$$H_p(fw_0,t)=\int_0^{+\infty}\frac{\cos\left(\log(x+6)\right)}{(x-t)^{p+1}}e^{-x}dx,\quad \alpha=0,\quad p=0,1.$$

In this case $f(x)=\cos\left(\log(x+6)\right)$ is very smooth and, then, according to our theoretical expectation the convergence is fast. In fact, looking at Table 5, we get errors of the order of the machine precision taking $m=70$ but only $j=48$ and $q=35$.

m	j	q	$\bar{\rho}_{0,m}(f, 0.1)$	$\bar{\rho}_{0,m}(f, 0.2)$	$\bar{\rho}_{0,m}(f, 3.5)$	$\bar{\rho}_{0,m}(f, 10)$
100	100	81	$1.0555e-4$	$5.6316e-5$	$9.8662e-6$	$2.5275e-5$
200	171	129	$4.3325e-8$	$1.1087e-7$	$9.5529e-8$	$1.2789e-7$
400	200	146	$8.3446e-10$	$4.9740e-10$	$3.4065e-11$	$2.6675e-11$
600	211	155	$2.8262e-12$	$1.5877e-12$	$5.8857e-14$	$4.6367e-14$
800	328	235	$1.4909e-14$	$7.3510e-15$	$3.8399e-16$	$2.3319e-15$
900	392	280	$-$	$-$	$-$	$-$
m	j	q	$\bar{\rho}_{1,m}(f, 0.1)$	$\bar{\rho}_{1,m}(f, 0.2)$	$\bar{\rho}_{1,m}(f, 3.5)$	$\bar{\rho}_{1,m}(f, 10)$
100	100	81	$3.2158e-3$	$4.2175e-4$	$1.5074e-5$	$9.3792e-5$
200	171	129	$1.5125e-6$	$1.7646e-6$	$1.7951e-8$	$1.4990e-6$
400	200	146	$3.8535e-8$	$1.0443e-8$	$2.6631e-11$	$5.3770e-10$
600	211	155	$7.4388e-11$	$3.8818e-11$	$1.2317e-12$	$2.2854e-12$
800	328	235	$6.2526e-13$	$3.5572e-13$	$3.4707e-15$	$4.4118e-15$
900	392	280	$2.8221e-14$	$6.1863e-15$	$7.6747e-16$	$4.6303e-15$
m	j	q	$\bar{\rho}_{2,m}(f, 0.1)$	$\bar{\rho}_{2,m}(f, 0.2)$	$\bar{\rho}_{2,m}(f, 3.5)$	$\bar{\rho}_{2,m}(f, 10)$
100	100	81	$1.4603e-2$	$1.2033e-2$	$1.3999e-4$	$1.5258e-4$
200	171	129	$5.5863e-5$	$6.3029e-4$	$2.7239e-6$	$8.4484e-7$
400	200	146	$1.2649e-6$	$5.5080e-7$	$1.9504e-9$	$6.7507e-10$
600	211	155	$7.6967e-9$	$2.1868e-9$	$4.7967e-12$	$2.0018e-12$
800	328	235	$6.7035e-11$	$1.2529e-11$	$4.3429e-14$	$9.4594e-14$
900	392	280	$1.3174e-12$	$2.5990e-14$	$1.4657e-14$	$7.8769e-16$

Table 3: Example 5.2: $\bar{\rho}_{p,m}(f, t)$, $p = 0, 1, 2$, with $t = 0.1, 0.2, 3.5, 10$

The graphical behavior of the functions $H^{(1)}_{p,70}(f, t), p = 0, 1$, is shown in Figure 5.

6. The proofs

Proof of Lemma 2.1. We first prove that $f^{(p)} \in Z_\lambda(u\varphi^p)$ implies $f \in Z_{\lambda+p}(u)$. Since

$$\sup_{s>0} \frac{\Omega_\varphi^{r+p}(f, s)_u}{s^{\lambda+p}} \quad \leq \quad \mathcal{C} \sup_{s>0} \frac{s^p\ \Omega_\varphi^r(f^{(p)}, s)_{u\varphi^p}}{s^{\lambda+p}} = \mathcal{C} \sup_{s>0} \frac{\Omega_\varphi^r(f^{(p)}, s)_{u\varphi^p}}{s^\lambda} < +\infty,$$

then

$$\|fu\| + \sup_{s>0} \frac{\Omega_\varphi^k(f, s)_u}{s^{\lambda+p}} < +\infty, \quad k > \lambda + p,$$

i.e., $f \in Z_{\lambda+p}(u)$. Now we prove that assuming $f \in Z_{\lambda+p}(u)$, then $f^{(p)} \in Z_\lambda(u\varphi^p)$. Since $f \in Z_{\lambda+p}(u)$, it is $\lim_m \omega_\varphi^r \left(f, \frac{1}{\sqrt{m}}\right)_u = 0$, i.e. $f \in C_u$ and therefore there exists a sequence of polynomials of best approximation $\{P_m\}_m$, such that

$$\lim_m \|(f - P_m)u\| = \lim_m \left\| \left(\sum_{k=0}^{\infty} (P_{2^{k+1}m} - P_{2^k m}) \right) u \right\| = 0.$$

Applying the Bernstein and the weak-Jackson inequalities [3, (3.4) and (3.7)]

$$\|(P_{2^{k+1}m} - P_{2^k m})^{(p)} u\varphi^p\| \leq \mathcal{C} \left(\sqrt{2^{k+1}m}\right)^p \|(P_{2^{k+1}m} - P_{2^k m})u\|$$

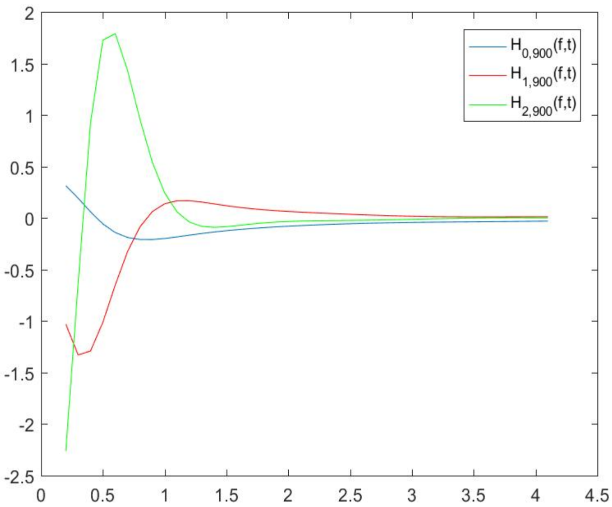


Figure 3: Example 2

m	j	q	$\bar{\rho}_{0,m}\,(f,0.25)$	$\bar{\rho}_{0,m}\,(f,2)$	$\bar{\rho}_{0,m}\,(f,3.5)$	$\bar{\rho}_{0,m}\,(f,7)$
100	96	77	$6.8668e-7$	$5.5415e-8$	$7.2331e-9$	$1.1894e-8$
200	150	112	$1.3653e-6$	$1.7428e-9$	$6.3430e-10$	$7.6955e-10$
400	215	159	$9.5389e-9$	$5.8301e-10$	$4.6992e-10$	$7.8659e-10$
600	269	197	$6.0573e-9$	$2.7446e-10$	$1.7752e-10$	$1.9522e-10$
800	312	228	$3.4956e-9$	$3.3817e-10$	$3.2248e-11$	$4.1977e-11$
m	j	q	$\bar{\rho}_{1,m}\,(f,0.25)$	$\bar{\rho}_{1,m}\,(f,2)$	$\bar{\rho}_{1,m}\,(f,3.5)$	$\bar{\rho}_{1,m}\,(f,7)$
100	96	77	$3.2608e-5$	$1.1232e-6$	$4.7436e-7$	$3.3086e-7$
200	150	112	$1.5876e-5$	$2.6713e-8$	$1.7878e-8$	$1.4129e-8$
400	215	159	$3.8635e-7$	$3.3807e-8$	$1.1751e-8$	$4.9654e-9$
600	269	197	$3.9030e-7$	$7.6649e-9$	$1.4749e-9$	$1.3910e-9$
800	312	228	$1.1754e-6$	$3.4733e-10$	$4.4317e-10$	$9.4341e-10$

Table 4: Example 5.3: $\bar{\rho}_{p,m}\,(f,t)\,,\ p=0,1,$ with $t=0.25,2,3.5,7$

$$\begin{aligned}
&\le \ \mathcal{C}\left(\sqrt{2^{k+1}m}\right)^{p} E_{2^k m}(f)_u \le \mathcal{C}\left(\sqrt{2^{k+1}m}\right)^{p}\int_0^{\frac{1}{\sqrt{m2^k}}}\frac{\Omega_\varphi^r(f,s)_u}{s}ds\\
&\le \ \mathcal{C}\left(\frac{1}{\sqrt{m2^k}}\right)^{\lambda}\sup_{s>0}\frac{\Omega_\varphi^r(f,s)_u}{s^{\lambda+p}},
\end{aligned}$$

where $\mathcal{C}\neq\mathcal{C}(m)$. Thus, by the assumption on f, we have

$$\begin{aligned}
E_m(f^{(p)})_{u\varphi^p} \ &\le \ \|(f-P_m)^{(p)}u\varphi^p\| = \sum_{k=0}^{\infty}\|(P_{2^{k+1}m}-P_{2^k m})^{(p)}u\varphi^p\|\\
&\le \ \frac{\mathcal{C}}{(\sqrt{m})^{\lambda}}\sum_{k=0}^{\infty}\frac{1}{(\sqrt{2^k})^{\lambda}} \le \frac{\mathcal{C}}{(\sqrt{m})^{\lambda}},
\end{aligned}$$

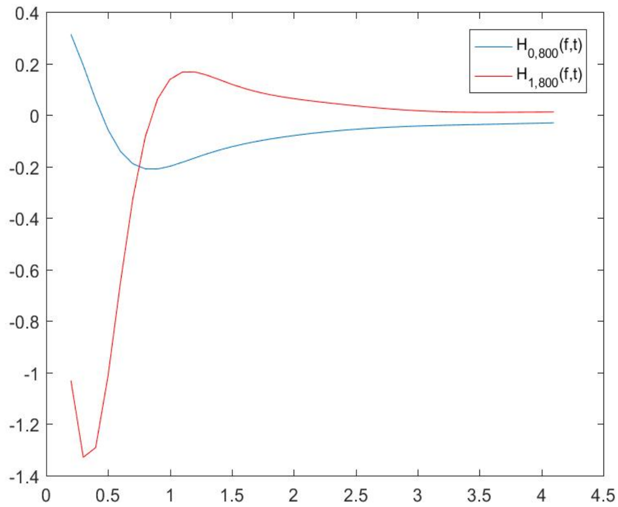


Figure 4: Example 3

where $\mathcal{C} \neq \mathcal{C}(m)$. Therefore, using the Salem-Stechkin inequality [3, (3.6)]

$$\sup_{s>0} \frac{\Omega_\varphi^r(f^{(p)}, s)_{u\varphi^p}}{s^\lambda} \leq \mathcal{C} \sup_{m\geq 1} (\sqrt{m})^\lambda E_m(f^{(p)})_{u\varphi^p} < +\infty,$$

i.e., $f^{(p)} \in Z_\lambda(u\varphi^p)$. □

Proof of Theorem 2.2. Let $P \in \mathcal{P}_m^*$. Using [13, Lemma 3.6] (see also [14, Lemma 2.1]) and the Bernstein inequality and by [11, Theorem 2.2] under the assumptions (7), we get

$$\begin{aligned} \|(f - L_{m+1}(w, f))^{(k)}\varphi^k u\| &\leq \mathcal{C}\left(\|(f-P)^{(k)}u\varphi^k\| + m^{\frac{k}{2}}\|L_{m+1}(w, f-P)u\|\right) \\ &\leq \mathcal{C}\left(m^{\frac{k}{2}} \log m \; E_M(f)_u + e^{-Am}\|fu\| + E_{m-k}(f^{(k)})_{u\varphi^k}\right). \end{aligned}$$

Recalling (4) and by the weak-Jackson inequality [3, (3.7)]

$$E_m(f)_u \leq \mathcal{C} \int_0^{\frac{1}{\sqrt{m}}} \frac{\Omega_\varphi^r(f,t)_u}{t} dt,$$

the theorem easily follows. □

Proof of Theorem 2.3. By arguments similar to those used in the proof of Theorem 2.2 we get

$$\|(f - L_{m+1,1}(w, f))^{(k)}\varphi^k u\| \leq \mathcal{C}\left(m^{\frac{k}{2}} \log m \; E_M(f)_u + E_{m-k}(f^{(k)})_{u\varphi^k} + e^{-Am}\|fu\|\right).$$

Therefore by (4)

$$\|(f - L_{m+1,1}(w, f))^{(k)}\varphi^k u\| \leq \mathcal{C}\left(\log m E_{M-k}(f^{(k)})_{u\varphi^k} + e^{-Am}\|fu\|\right) \tag{20}$$

and by (5) the thesis follows.

□

m	j	q	$\bar{\rho}^{(1)}_{0,m}(f,0.1)$	$\bar{\rho}^{(1)}_{0,m}(f,0.25)$	$\bar{\rho}^{(1)}_{0,m}(f,7)$	$\bar{\rho}^{(1)}_{0,m}(f,15)$
10	10	10	$2.6095e-10$	$2.6095e-9$	$6.7158e-7$	$2.3514e-5$
20	20	18	$1.7503e-12$	$9.1273e-12$	$1.3596e-9$	$2.1707e-8$
30	28	23	$3.2345e-14$	$7.6037e-16$	$1.6405e-11$	$9.1106e-10$
40	36	26	$6.1498e-16$	$1.0926e-15$	$8.9550e-14$	$1.2569e-11$
50	40	30	$-$	$-$	$3.0375e-15$	$3.8846e-13$
60	42	32	$-$	$-$	$-$	$9.2251e-15$
70	48	35	$-$	$-$	$-$	$-$
m	j	q	$\bar{\rho}^{(1)}_{1,m}(f,0.1)$	$\bar{\rho}^{(1)}_{1,m}(f,0.25)$	$\bar{\rho}^{(1)}_{1,m}(f,7)$	$\bar{\rho}^{(1)}_{1,m}(f,15)$
10	10	10	$6.4807e-10$	$2.8573e-8$	$2.1509e-7$	$3.3793e-6$
20	20	18	$7.3881e-11$	$1.6884e-11$	$1.7380e-9$	$1.2558e-7$
30	28	23	$6.4410e-13$	$9.9954e-13$	$1.2689e-11$	$1.3178e-10$
40	36	26	$5.3359e-15$	$1.6356e-14$	$5.3989e-13$	$7.5597e-12$
50	40	30	$-$	$-$	$1.7459e-14$	$8.1662e-14$
60	42	32	$-$	$-$	$5.2992e-16$	$2.5294e-14$
70	48	35	$-$	$-$	$-$	$8.0328e-16$

Table 5: Example 5.4: $\bar{\rho}_{p,m}(f,t)$, $p=0,1$, with $t=0.1,0.25,7,15$

In order to prove Theorem 3.1, we need some lemmas

Lemma 6.1. *For $\alpha \geq 0$, if $0<t\leq 1$ we have*

$$\left| \fint_{|x-t|<1} \frac{w(x)}{(x-t)^{p+1}} dx \right| \leq \mathcal{C} w(t) \begin{cases} t^{-p} & p \geq 1 \\ \log t^{-1} & p=0 \end{cases},$$

and if $t>1$ we get

$$\left| \fint_{|x-t|<1} \frac{w(x)}{(x-t)^{p+1}} dx \right| \leq \mathcal{C} w(t),$$

where in both the cases $0<\mathcal{C}\neq\mathcal{C}(t)$.

Proof. Set

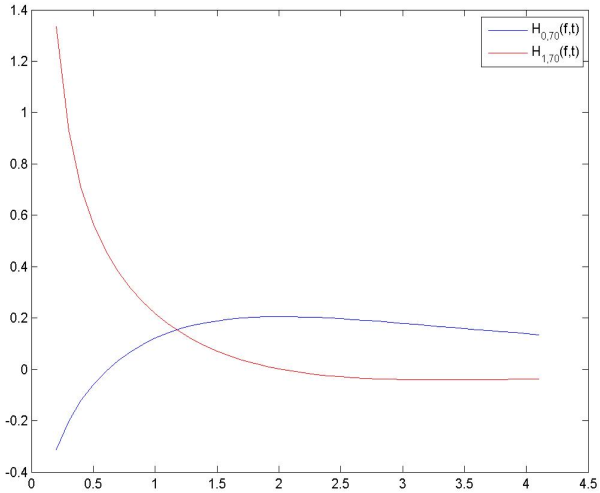
$$M_p(t) = \fint_{|x-t|<1} \frac{w(x)}{(x-t)^{p+1}} dx.$$

Let assume $\alpha>0$, since the proof in case $\alpha=0$ is easier. At first consider $p=0$. For $0<t<1$

$$\begin{aligned} |M_0(t)| &\leq \left| \int_0^{2t} \frac{e^{-x}-e^{-t}}{(x-t)} x^\alpha dx \right| + e^{-t} \left| \int_0^{2t} \frac{x^\alpha - t^\alpha}{x-t} dx \right| + \left| \int_{2t}^{t+1} \frac{w(x)}{x-t} dx \right| \\ &\leq \mathcal{C} t^{\alpha+1} + e^{-t} \int_0^{2t} |x-t|^{\alpha-1} dx + w(2t) \int_{2t}^{t+1} \frac{dx}{x-t} \\ &\leq \mathcal{C} t^\alpha \log \frac{1}{t} \end{aligned}$$

and in the case $t>1$

$$\begin{aligned} |M_0(t)| &\leq \left| \int_{t-1}^{t+1} \frac{e^{-x}-e^{-t}}{(x-t)} x^\alpha dx \right| + e^{-t} \left| \fint_{t-1}^{t+1} \frac{x^\alpha}{x-t} dx \right| \\ &\leq \mathcal{C} e^{-t} \left(t^\alpha + \left| \int_{t-1}^{t+1} \frac{x^\alpha - t^\alpha}{x-t} dx \right| \right) \leq \mathcal{C} \left[w(t) + e^{-t} \int_{t-1}^{t+1} |x-t|^{\alpha-1} dx \right] \leq \mathcal{C} w(t). \end{aligned}$$



Assume now $p \geq 1$ and let $0 < t < 1$. We have

$$|M_p(t)| \leq \left| \fint_0^{2t} \frac{w(x)}{(x-t)^{p+1}} dx \right| + \left| \int_{2t}^{t+1} \frac{w(x)}{(x-t)^{p+1}} dx \right| =: |A(t)| + |B(t)|. \tag{21}$$

Starting from $A(t)$, we get

$$\begin{aligned} |A(t)| &\leq \left| \int_0^{2t} \frac{e^{-x} - e^{-t} \sum_{k=0}^{p} \frac{(-1)^k}{k!}(x-t)^k}{(x-t)^{p+1}} x^\alpha dx \right| + \sum_{k=0}^{p} \frac{e^{-t}}{k!} \left| \fint_0^{2t} \frac{x^\alpha}{(x-t)^{p-k+1}} dx \right| \\ &\leq \frac{1}{(p+1)!} \int_0^{2t} e^{-\xi} x^\alpha dx + \sum_{k=0}^{p} \frac{e^{-t}}{k!} \left| \fint_0^{2t} \frac{x^\alpha}{(x-t)^{p-k+1}} dx \right| \\ &\leq \mathcal{C} t^\alpha + \sum_{k=0}^{p-1} \frac{e^{-t}}{k!} \left| \widetilde{A}_k(t) \right|, \qquad \widetilde{A}_k = \fint_0^{2t} \frac{x^\alpha}{(x-t)^{p-k+1}} dx. \end{aligned} \tag{22}$$

Since

$$\fint_0^{2t} \frac{x^\alpha}{x-t} dx = \int_0^{2t} \frac{x^\alpha - t^\alpha}{x-t} dx = t^\alpha \int_{-1}^{1} \frac{(1+y)^\alpha - 1}{y} dy =: \mathcal{C}_0 t^\alpha$$

and, for any $i = 1, 2, \ldots$,

$$\begin{aligned} &\frac{d^i}{dt^i} \int_0^{2t} \frac{x^\alpha - t^\alpha}{x-t} dx = i! \int_0^{2t} \frac{x^\alpha - \sum_{s=0}^{i} d_s t^{\alpha-s} (x-t)^s}{(x-t)^{i+1}} dx - \mathcal{C}_1 t^{\alpha-i}, \\ &d_s = \frac{\alpha(\alpha-1)\ldots(\alpha-s+1)}{s!}, \quad \mathcal{C}_1 = \mathcal{C}_1(\alpha), \quad \mathcal{C}_1 \neq \mathcal{C}_1(t), \end{aligned}$$

we have

$$\begin{aligned} \widetilde{A}_k(t) &= \frac{1}{(p-k)!} \left[\frac{d^{p-k}}{dt^{p-k}} \int_0^{2t} \frac{x^\alpha - t^\alpha}{x-t} dx + \mathcal{C}_2 t^{\alpha-p+k} \right] + \sum_{s=0}^{p-k} d_s t^{\alpha-s} \fint_0^{2t} \frac{dx}{(x-t)^{p+1-k-s}} \\ &= \frac{1}{(p-k)!} \left[\frac{d^{p-k}}{dt^{p-k}} \mathcal{C}_0 t^\alpha + \mathcal{C}_2 t^{\alpha-p+k} \right] + \mathcal{C}_3 t^{\alpha-p+k} \end{aligned}$$

and then

$$\left|\widetilde{A}_k(t)\right| \quad \leq \quad \mathcal{C}t^{\alpha-p+k}.$$

Thus, combining last inequality with (22), we can conclude

$$|A(t)| \leq \mathcal{C}w(t)t^{-p}. \tag{23}$$

Since it can be easily deduced that

$$|B(t)| \leq \mathcal{C}w(t)t^{-p},$$

combining last inequality with (23) and (21), the lemma follows for $0 < t < 1$.

Assume now $t \geq 1$. Since

$$\begin{aligned} |M_p(t)| &\leq \left|\fint_{t-1}^{t+1} \frac{w(x)}{(x-t)^{p+1}}dx\right| \leq \frac{1}{(p+1)!}\left|\int_{t-1}^{t+1} e^{-\xi}x^{\alpha}dx\right| + \sum_{k=0}^{p}\frac{e^{-t}}{k!}\left|\fint_{t-1}^{t+1}\frac{x^{\alpha}}{(x-t)^{p+1-k}}dx\right| \\ &\leq \mathcal{C}e^{-t}t^{\alpha} + e^{-t}\sum_{k=0}^{p}\frac{1}{k!}|D_k(t)| \end{aligned}$$

and

$$|D_k(t)| \leq \mathcal{C}\left|\int_{t-1}^{t+1}\xi^{\alpha-p-1+k}dx\right| + \sum_{i=0}^{p-k} d_i t^{\alpha-i}\left|\fint_{t-1}^{t+1}\frac{dx}{(x-t)^{p+1-k-i}}\right| \leq \mathcal{C}t^{\alpha-p+k},$$

it follows that

$$|M_p(t)| \leq \mathcal{C}e^{-t}t^{\alpha-p} \leq \mathcal{C}w(t).$$

So the lemma is completely proved. □

Having set $R_p(f,x,t) = f(x) - \sum_{k=0}^{p}\frac{f^{(k)}(t)}{k!}(x-t)^k$, we recall the Peano form of the Taylor's remainder term

$$R_p(f,x,t) = \frac{1}{(p-1)!}\int_t^x [f^{(p)}(\tau) - f^{(p)}(t)](x-\tau)^{p-1}d\tau. \tag{24}$$

Lemma 6.2. *Let $0 \leq \gamma \leq \alpha$, $p \geq 0$ integer, $0 < \lambda < 1$ and $t > 0$. If $f^{(p)} \in Z_\lambda(u\varphi^p)$ we have*

$$\varphi^p(t)\left|\int_{|x-t|<1}\frac{R_p(f,x,t)}{(x-t)^{p+1}}w(x)dx\right| \quad \leq \quad \mathcal{C}\left(\int_0^1\frac{\Omega_\varphi(f^{(p)},\sigma)_{u\varphi^p}}{\sigma}d\sigma + \psi(t)\|f\|_{W_p(u)}\right),$$

where

$$\psi(t) = \begin{cases} \log t^{-1}, & \alpha = \gamma \text{ and } 0 < t < 1 \\ 1, & \text{otherwise} \end{cases}$$

and $0 < \mathcal{C} \neq \mathcal{C}(t,f)$.

Proof. We first assume $0 < t < 1$ and use the following decomposition

$$\begin{aligned} \int_{|x-t|<1}\frac{R_p(f,x,t)}{(x-t)^{p+1}}w(x)dx &= \left\{\int_0^{2t} + \int_{2t}^{t+1}\right\}\frac{R_p(f,x,t)}{(x-t)^{p+1}}w(x)dx \\ &=: I_1(t) + I_2(t). \end{aligned} \tag{25}$$

Using (24) we get

$$\begin{aligned} I_1(t) &= \frac{1}{(p-1)!}\int_0^t\left[\int_x^t [f^{(p)}(t) - f^{(p)}(\tau)](\tau-x)^{p-1}d\tau\right]\frac{w(x)}{(t-x)^{p+1}}dx \\ &+ \frac{1}{(p-1)!}\int_t^{2t}\left[\int_t^x [f^{(p)}(\tau) - f^{(p)}(t)](x-\tau)^{p-1}d\tau\right]\frac{w(x)}{(x-t)^{p+1}}dx \end{aligned}$$

and, by the changes of variables $x = t - \sigma\sqrt{t}$, $\tau = t - z\sqrt{t}$ in the first integral and $x = t + \sigma\sqrt{t}$,$\tau = t + z\sqrt{t}$ in the second integral, we get

$$\begin{aligned} I_1(t) &= \frac{1}{(p-1)!}\int_0^{\sqrt{t}}\left[\int_0^{\sigma}[f^{(p)}(t) - f^{(p)}(t - z\sqrt{t})](\sigma - z)^{p-1}dz\right]\frac{w(t-\sigma\sqrt{t})}{\sigma^{p+1}}d\sigma \\ &+ \frac{1}{(p-1)!}\int_0^{\sqrt{t}}\left[\int_0^{\sigma}[f^{(p)}(t + z\sqrt{t}) - f^{(p)}(t)](\sigma - z)^{p-1}dz\right]\frac{w(t+\sigma\sqrt{t})}{\sigma^{p+1}}d\sigma. \end{aligned}$$

Thus, we obtain

$$|I_1(t)| \leq \mathcal{C}\int_0^{\sqrt{t}}\frac{\Omega_\varphi(f^{(p)},\sigma)_{u\varphi^p}}{\sigma}\left[\frac{w(t-\sigma\sqrt{t})}{u(t)\varphi^p(t)} + \frac{w(t+\sigma\sqrt{t})}{u(t)\varphi^p(t)}\right]d\sigma$$

by which

$$|I_1(t)|\varphi^p(t) \leq \mathcal{C}e^{-\frac{t}{2}}t^{\alpha-\gamma}\int_0^1\frac{\Omega_\varphi(f^{(p)},\sigma)_{u\varphi^p}}{\sigma}d\sigma. \tag{26}$$

By (24), for $x > t$,

$$|R_p(f,x,t)| \leq \frac{1}{(p-1)!}\int_t^x|f^{(p)}(\tau) - f^{(p)}(t)|(x-\tau)^{p-1}d\tau \leq \mathcal{C}\|f^{(p)}\varphi^p u\|\frac{(x-t)^p}{u(x)\varphi^p(t)}$$

and therefore

$$|I_2(t)| \leq \mathcal{C}\frac{\|f^{(p)}\varphi^p u\|}{\varphi^p(t)}\int_{2t}^{t+1}\frac{w(x)}{u(x)(x-t)}dx$$

Then, since $x - t > \frac{x}{2}$ we have

$$\begin{aligned} |I_2(t)|\varphi^p(t) &\leq \mathcal{C}\|f^{(p)}\varphi^p u\|e^{-t}\begin{cases}\int_0^2 x^{\alpha-\gamma-1}dx, & \alpha > \gamma \\ \log t^{-1}, & \alpha = \gamma\end{cases} \\ &\leq \mathcal{C}\|f^{(p)}\varphi^p u\|\psi(t). \end{aligned} \tag{27}$$

Combining (26) and (27) with (25) the thesis follows for $0 < t < 1$.

In the case $t \geq 1$, by similar arguments used in the previous case, we get

$$\left|\int_{t-1}^{t+1}\frac{R_p(f,x,t)}{(x-t)^{p+1}}w(x)dx\right| \leq \frac{\mathcal{C}}{\varphi^p(t)}\int_0^1\frac{\Omega_\varphi(f^{(p)},\sigma)_{u\varphi^p}}{\sigma}d\sigma,$$

which completes the proof. □

Lemma 6.3. *Let $0 < \lambda < 1$, $0 < \gamma \leq \alpha$ and $s \in I\!N_0$. If $f \in Z_{s+\lambda}(u)$ then $\mathbf{F}(f) \in W_s(u)$ and*

$$\|\mathbf{F}^{(s)}(f)\varphi^s u\| \leq \mathcal{C}\|f\|_{Z_{s+\lambda}(u)}, \quad 0 < \mathcal{C} \neq \mathcal{C}(m,f). \tag{28}$$

Proof. First we prove that $F(f) \in C_u$ and estimate (28) for $s = 0$. Start from

$$\mathbf{F}(f,t) = \left(\int_{|x-t|>1} + \int_{|x-t|<1}\right)\frac{f\ (x) - f\ (t)}{x-t}w(x)dx =: A(t) + B(t). \tag{29}$$

We have

$$u(t)|A(t)| \leq \mathcal{C}u(t)\left\{\|fu\| + |f(t)|\int_0^{+\infty}w(x)dx\right\} \tag{30}$$

and by Lemma 6.2 for $p=0$, which holds under the assumption $\gamma \le \alpha$, we get

$$u(t)|B(t)| \quad \le \quad \mathcal{C}u(t)\left(\int_0^1 \frac{\Omega_\varphi(f,\sigma)_u}{\sigma}d\sigma + \psi(t)\|fu\|\right). \tag{31}$$

Consequently, combining (30) and (31) with (29), under the condition $\gamma > 0$ we deduce

$$\lim_{t\to 0^+} \mathbf{F}(f,t)u(t) = 0$$

and

$$|\mathbf{F}(f,t)u(t)| \le \mathcal{C}\|f\|_{Z_\lambda(u)}.$$

Similarly we proceed for $t \ge 1$, obtaining

$$|\mathbf{F}(f,t)u(t)| \le \mathcal{C}\|f\|_{Z_\lambda(u)}, \quad \lim_{t\to+\infty} \mathbf{F}(f,t)u(t) = 0,$$

Thus we can conclude $\mathbf{F}(f) \in C_u$.

Now we prove (28) with $s \ge 1$. With $R_s(f,x,t)$ in (24),

$$\mathbf{F}^{(s)}(f,t) = \left\{\int_{|x-t|<1} + \int_{|x-t|\ge 1}\right\} \frac{R_s(f,x,t)}{(x-t)^{s+1}} w(x)dx =: A_1(t) + A_2(t) \tag{32}$$

For any $t > 0$

$$\begin{aligned}\varphi^s(t)u(t)|A_2(t)| &= \varphi^s(t)u(t)\left|\int_{t+1}^{+\infty} \frac{R_s(f,x,t)}{(x-t)^{s+1}} w(x)dx\right| \\ &\le \mathcal{C}\left\{\|fu\|\varphi^s(t)u(t) + \mathcal{C}\sum_{k=0}^{s} \frac{\|f^{(k)}\varphi^k u\|}{k!}\varphi^{s-k}(t)\right\} e^{-(t+1)/2}\int_{t+1}^{+\infty} e^{-\frac{x}{2}} x^\alpha(x)dx,\end{aligned}$$

and taking into account that for any function f s.t. $\|f^{(j)}\varphi^{(j)}u\| < \infty$, [4, p. 310, Lemma 2.1]

$$\sum_{j=0}^{s} a_j \|f^{(j)}\varphi^{(j)}u\| \le \mathcal{C}\left(\|fu\|_\infty + \|f^{(s)}\varphi^{(s)}u\|\right), \quad a_j \text{ positive constants},$$

it follows that

$$\varphi^s(t)u(t)|A_2(t)| \le \mathcal{C}\left\{\|fu\| + \|f^{(s)}\varphi^s u\|\right\}.$$

In the case $t > 1$, we have the additional integral

$$\begin{aligned}\varphi^s(t)u(t)\left|\int_0^{t-1} \frac{R_s(f,x,t)}{(x-t)^{s+1}} w(x)dx\right| &\le \mathcal{C}\left\{\|fu\|\varphi^s(t)u(t) + \sum_{k=0}^{s} \frac{\|f^{(k)}\varphi^k u\|}{k!}\varphi^{s-k}(t)\int_0^{t-1} \frac{dx}{(x-t)^{s+1-k}}\right\} \\ \le \mathcal{C} \quad &\left\{\|fu\|\varphi^s(t)u(t) + \sum_{k=0}^{s} \frac{\|f^{(k)}\varphi^k u\|}{k!} \frac{1}{\varphi^{s-k}(t)}\right\} \le \mathcal{C}\left\{\|fu\| + \|f^{(s)}\varphi^s u\|\right\},\end{aligned}$$

being $\alpha \ge 0$. Thus, in all the cases we get

$$\varphi^s(t)u(t)|A_2(t)| \le \mathcal{C}\left\{\|fu\| + \|f^{(s)}\varphi^s u\|\right\}. \tag{33}$$

Moreover, by Lemma 6.2 and Lemma 2.1, we get

$$\varphi^s(t)u(t)|A_1(t)| \le \mathcal{C}\|f\|_{Z_{s+\lambda}(u)}. \tag{34}$$

Finally, combining (34) and (33) with (32), (28) follows.

□

Lemma 6.4. *Let $0 \le \gamma \le \alpha$, $0 < \lambda < 1$. If $f \in Z_{s+\lambda}(u)$ with $s \in I\!N_0$, then for any fixed m, $\mathbf{F}_m \in W_s(u)$ and*

$$\|\mathbf{F}_m^{(s)}(f)\varphi^s u\| \le \mathcal{C} \log m \|f\|_{Z_{s+\lambda}(u)}, \quad \mathcal{C} \ne \mathcal{C}(m, f) \tag{35}$$

Proof. To prove (35) with $s = 0$ we start from

$$\mathbf{F}_m(f,t) = \left(\sum_{|x_{m,k}-t|\ge 1} + \sum_{|x_{m,k}-t|<1} \right) \frac{f\ (x_{m,k}) - f\ (t)}{x_{m,k} - t} \lambda_{m,k} =: \Sigma_1(t) + \Sigma_2(t). \tag{36}$$

We have

$$\begin{aligned} u(t)|\Sigma_1(t)| &\le \mathcal{C}u(t)\|fu\| \sum_{k=1}^{j} \Delta x_{m,k} \frac{w(x_{m,k})}{u(x_{m,k})} + |f(t)|u(t) \sum_{k=1}^{j} \lambda_{m,k} \\ &\le \mathcal{C}\|fu\|, \end{aligned} \tag{37}$$

being [13] $\lambda_{m,k} \sim \Delta x_{m,k} w(x_{m,k}), k = 1, 2, \dots m-1$, with $\Delta x_{m,k} = x_{m,k+1} - x_{m,k}$. Denoted by $x_{m,d}$ the closest knot to t, we have

$$\begin{aligned} u(t)|\Sigma_2(t)| &\le \mathcal{C}\|fu\| \sum_{\substack{|x_{m,k}-t|<1 \\ k\ne d}} \frac{\lambda_{m,k}}{|x_{m,k}-t|} \frac{u(t)}{u(x_{m,k})} + |f(t)|u(t) \sum_{\substack{|x_{m,k}-t|<1 \\ k\ne d}} \frac{\lambda_{m,k}}{|x_{m,k}-t|} \\ &+ u(t) \frac{|f(x_{m,d})|}{|x_{m,d}-t|} \lambda_{m,d} + |f(t)|u(t) \frac{\lambda_{m,d}}{|x_{m,d}-t|} =: \sum_{i=1}^{4} S_i(t). \end{aligned}$$

To estimate S_1 and S_2 we use an argument in the proof of [2, Th. 3.1, p. 223], to obtain under the assumption $\alpha \ge \gamma$

$$S_1(t) + S_2(t) \le \|fu\| w(t) \log m.$$

Taking into account $t \sim x_{m,d}$, $\Delta x_{m,d} \sim |x_{m,d} - t|$, [3, Lemma 4.1] $w(x_{m,d}) \sim w(t)$, we get

$$S_3(t) + S_4(t) \le \mathcal{C}\|fu\| w(t). \tag{38}$$

(35) with $s = 0$ follows by combining (37)-(38) with (36). Now we prove (35) with $s \ge 1$. Let $x_{m,d} < t < x_{m,d+1}$, being $x_{m,d}$ the zero of $p_m(w_\alpha)$ closest to t. Then, we will use the following decomposition

$$\begin{aligned} \mathbf{F}_m^{(s)}(fw, t) &= \left\{ \sum_{k=1}^{d-1} + \sum_{k=d}^{d+1} + \sum_{k=d+2}^{j} \right\} \frac{f(x_k) - \sum_{j=0}^{s} \frac{f^{(j)}(t)}{j!}(x_k - t)^j}{(x_k - t)^{s+1}} \lambda_{m,k} \\ &=: A(t) + B(t) + C(t). \end{aligned} \tag{39}$$

Now we estimate $A(t)$. Using (24), since $x_k \le x_d < t$, we have

$$\begin{aligned} \left| f(x_k) - \sum_{j=0}^{s} \frac{f^{(j)}(t)}{j!}(x_k - t)^j \right| &\le \frac{1}{(s-1)!} \int_{x_k}^{t} |f^{(s)}(\tau) - f^{(s)}(t)| (\tau - x_k)^{s-1} d\tau \\ &\le \mathcal{C}\|f^{(s)}\varphi^s u\| \int_{x_k}^{t} \frac{(\tau - x_k)^{s-1}}{\varphi^s(\tau)u(\tau)} d\tau \\ &+ \mathcal{C} \frac{\|f^{(s)}\varphi^s u\|}{\varphi^s(t)u(t)} \int_{x_k}^{t} (\tau - x_k)^{s-1} d\tau \le \mathcal{C}\|f^{(s)}\varphi^s u\| \frac{(t - x_k)^s}{\varphi^s(t)u(t)}. \end{aligned}$$

Then, since (see [2, (5.25)])

$$\sum_{k=1}^{d-1} \frac{\lambda_{m,k}}{(t-x_k)} \leq \int_0^{x_d} \frac{x^\alpha e^{-x}}{(t-x)} dx$$

and $\frac{t-x_d}{t} \sim 1$, we obtain

$$\begin{aligned} |A(t)| &\leq \mathcal{C} \frac{\|f^{(s)}\varphi^s u\|}{\varphi^s(t)u(t)} \sum_{k=1}^{d-1} \frac{\lambda_{m,k}}{(t-x_k)} \leq \mathcal{C} \frac{\|f^{(s)}\varphi^s u\|}{\varphi^s(t)u(t)} \int_0^{x_d} \frac{x^\alpha e^{-x}}{(t-x)} dx \\ &\leq \mathcal{C} \frac{\|f^{(s)}\varphi^s u\|}{\varphi^s(t)u(t)} \log \frac{t-x_d}{t} \leq \mathcal{C} \frac{\|f^{(s)}\varphi^s u\|}{\varphi^s(t)u(t)}. \end{aligned} \tag{40}$$

In order to estimate $B(t)$, making the change of variable $\tau = t - z\sqrt{t}$, we get

$$\left| f(x_d) - \sum_{j=0}^{s} \frac{f^{(j)}(t)}{j!}(x_d - t)^j \right| = \frac{\sqrt{t}}{(s-1)!} \int_0^{\frac{t-x_d}{\sqrt{t}}} |f^{(s)}(t) - f^{(s)}(t - z\sqrt{t})|(t - x_d - z\sqrt{t})^{s-1} dz$$

$$= \frac{\sqrt{t}}{(s-1)!\varphi^s(t)u(t)} \int_0^{\frac{t-x_d}{\sqrt{t}}} |\Delta_{z\varphi} f^{(s)}(t)\varphi^s(t)u(t)|(t - x_d - z\sqrt{t})^{s-1} dz.$$

Since $\Delta x_{m,k} \sim \frac{\sqrt{x_{m,k}}}{\sqrt{m}}, k = 1, 2, \ldots, j$, it results $t - x_d \sim \Delta x_d \sim \frac{\sqrt{t}}{\sqrt{m}}$ and, then

$$\begin{aligned} \left| f(x_d) - \sum_{j=0}^{s} \frac{f^{(j)}(t)}{j!}(x_d - t)^j \right| &\leq \frac{\mathcal{C}}{\varphi^{s-1}(t)u(t)} \sup_{0<z\frac{1}{\sqrt{m}}} \|(\Delta_{z\varphi} f^{(s)})\varphi^s u\| \int_0^{\frac{(t-x_d)}{\sqrt{t}}} (t - x_d - z\sqrt{t})^{s-1} dz \\ &\leq \frac{\mathcal{C}}{\varphi^{s-1}(t)u(t)} \Omega_\varphi \left(f^{(s)}, \frac{1}{\sqrt{m}} \right)_{u\varphi^s} (t - x_d)^s. \end{aligned}$$

Therefore,

$$\frac{\left| f(x_d) - \sum_{j=0}^{s} \frac{f^{(j)}(t)}{j!}(x_d - t)^j \right| \lambda_{m,d}}{(t - x_d)^{s+1}} \leq \mathcal{C} \frac{w(x_d)}{\varphi^{s-1}(t)u(t)} \Omega_\varphi \left(f^{(s)}, \frac{1}{\sqrt{m}} \right)_{u\varphi^s} \frac{\Delta x_d}{(t - x_d)}$$

and by

$$\Omega_\varphi \left(f^{(s)}, \frac{1}{\sqrt{m}} \right)_{u\varphi^s} \leq \mathcal{C} \int_0^{\frac{1}{\sqrt{m}}} \frac{\Omega_\varphi \left(f^{(s)}, t \right)_{u\varphi^s}}{t} dt,$$

under the assumption on γ, we obtain

$$\frac{\left| f(x_d) - \sum_{j=0}^{s} \frac{f^{(j)}(t)}{j!}(x_d - t)^j \right| \lambda_{m,d}}{(t - x_d)^{s+1}} \leq \mathcal{C} t^{\alpha - \gamma - \frac{s-1}{2}} e^{-\frac{t}{2}} \int_0^{\frac{1}{\sqrt{m}}} \frac{\Omega_\varphi \left(f^{(s)}, t \right)_{u\varphi^s}}{t} dt.$$

Since last estimate holds replacing x_d with x_{d+1}, we conclude

$$|B(t)| \leq \mathcal{C} t^{\alpha - \gamma - \frac{s-1}{2}} e^{-\frac{t}{2}} \int_0^{\frac{1}{\sqrt{m}}} \frac{\Omega_\varphi \left(f^{(s)}, t \right)_{u\varphi^s}}{t} dt$$

and therefore

$$|B(t)|u(t)\varphi^s(t) \leq \mathcal{C} t^{\alpha + \frac{1}{2}} e^{-t} \int_0^{\frac{1}{\sqrt{m}}} \frac{\Omega_\varphi \left(f^{(s)}, t \right)_{u\varphi^s}}{t} dt. \tag{41}$$

It remains to estimate $C(t)$. For $t < x_{d+2} \le x_k$, we have

$$\begin{aligned}
\left| f(x_k) - \sum_{j=0}^{s} \frac{f^{(j)}(t)}{j!}(x_k - t)^j \right| &\le \frac{1}{(s-1)!}\int_t^{x_k} |f^{(s)}(\tau) - f^{(s)}(t)|(x_k - \tau)^{s-1} d\tau \\
&\le \mathcal{C}\|f^{(s)}\varphi^s u\| \int_t^{x_k} \frac{(x_k-\tau)^{s-1}}{\varphi^s(\tau)u(\tau)} d\tau + \mathcal{C}\frac{\|f^{(s)}\varphi^s u\|}{\varphi^s(t)u(t)} \int_t^{x_k} (x_k - \tau)^{s-1} d\tau \\
&\le \mathcal{C}\|f^{(s)}\varphi^s u\| \frac{(x_k - t)^s e^{\frac{x_k}{2}}}{t^{\gamma+\frac{s}{2}}} + \mathcal{C}\|f^{(s)}\varphi^s u\| \frac{(x_k-t)^s}{\varphi^s(t)u(t)}.
\end{aligned}$$

Then, for $\alpha \ge 0$,

$$\begin{aligned}
|C(t)| &\le \mathcal{C}\frac{\|f^{(s)}\varphi^s u\|}{t^{\gamma+\frac{s}{2}}} \sum_{k=d+2}^{j} \frac{\Delta x_k x_k^\alpha e^{-\frac{x_k}{2}}}{(x_k - t)} + \mathcal{C}\frac{\|f^{(s)}\varphi^s u\|}{\varphi^s(t)u(t)} \sum_{k=d+2}^{j} \frac{\lambda_{m,k}}{(x_k - t)} \\
&\le \mathcal{C}\frac{\|f^{(s)}\varphi^s u\|}{t^{\gamma+\frac{s}{2}}} \int_{x_{d+1}}^{x_j} \frac{x^\alpha e^{-\frac{x}{2}}}{(x-t)} dx + \mathcal{C}\frac{\|f^{(s)}\varphi^s u\|}{\varphi^s(t)u(t)} \int_{x_{d+1}}^{x_j} \frac{x^\alpha e^{-x}}{(x-t)} dx \\
&\le \mathcal{C}\frac{\|f^{(s)}\varphi^s u\|}{t^{\gamma+\frac{s}{2}}} \log\frac{x_j - t}{x_{d+1} - t} + \mathcal{C}\frac{\|f^{(s)}\varphi^s u\|}{\varphi^s(t)u(t)} \log\frac{x_j - t}{x_{d+1} - t} \\
&\le \mathcal{C}\frac{\|f^{(s)}\varphi^s u\|}{t^{\gamma+\frac{s}{2}}} \log m + \mathcal{C}\frac{\|f^{(s)}\varphi^s u\|}{\varphi^s(t)u(t)} \log m. \qquad (42)
\end{aligned}$$

The thesis follows by combining (40), (41) and (42) with (39). □

Proof of Theorem 3.1. We consider only the case $p \ge 1$, being the case $p = 0$ simpler. We first prove (14). Start from

$$\begin{aligned}
|\rho_{p,m}(f,t)|\varphi^p(t)u(t) &\le \|[\mathbf{F}(f) - L_{m+1}(w, \mathbf{F}(f))]^{(p)}\varphi^p u\| \\
&+ \|L_{m+1}(w, \mathbf{F}(f) - \mathbf{F}_m(f))^{(p)}\varphi^p u\| =: S_1 + S_2. \qquad (43)
\end{aligned}$$

Since $f \in Z_{p+r+\lambda}(u)$, by Lemma 6.3 it follows $\mathbf{F}(f) \in W_{p+r}(u)$. Then, by (8) and (28)

$$S_1 \le \frac{\mathcal{C}}{(\sqrt{m})^r} \log m \|\mathbf{F}(f)\|_{W_{p+r}(u)} \le \frac{\mathcal{C}}{(\sqrt{m})^r} \log m \|f\|_{Z_{p+r+\lambda}(u)}. \qquad (44)$$

By the Bernstein inequality [3, (3.4)] and [11, Theorem 2.2]

$$S_2 \le \mathcal{C}\sqrt{m^p}\|L_{m+1}(w, \mathbf{F}(f) - \mathbf{F}_m(f))u\| \le \mathcal{C}\sqrt{m^p}\log m[E_M(\mathbf{F}(f))_u + E_M(\mathbf{F}_m(f))_u].$$

Using Lemmas 6.3 and 6.4 together with (4) we deduce

$$S_2 \le \frac{\mathcal{C}}{(\sqrt{m})^r} \log^2 m \|f\|_{Z_{p+r+\lambda}(u)}.$$

(14) follows combining last estimate and (44) with (43).

Now we prove (13). Since

$$\|L_{m+1}^{(p)}(w, \mathbf{F}_m(f))\varphi^p u\| \le \|(\mathbf{F}_m(f) - L_{m+1}(w, \mathbf{F}_m(f)))^{(p)}\varphi^p u\| + \|\mathbf{F}_m^{(p)}(f)\varphi^p u\|,$$

by (8) and (35), (13) follows. □

Proof of Theorem 3.2. We consider only the case $p \geq 1$, being the case $p = 0$ simpler. We first prove (17). Start from

$$\begin{aligned} |\rho^{(1)}_{p,m}(f,t)|\varphi^p(t)u(t) &\leq \|[\mathbf{F}(f) - L_{m+1,1}(w, \mathbf{F}(f))]^{(p)}\varphi^p u\| \\ &+ \|L_{m+1,1}(w, \mathbf{F}(f) - \mathbf{F}_m(f))^{(p)}\varphi^p u\| =: T_1 + T_2. \end{aligned} \tag{45}$$

By (20) and taking into account that [3, p. 189]

$$E_n(f^{(k)})_{u\varphi^k} \leq \mathcal{C} \sup_n n^{\frac{k}{2}} E_n(f)_u,$$

we get

$$T_1 \leq \mathcal{C} \log m E_{M-p}(\mathbf{F}(f)^{(p)})_{u\varphi^p} \leq \mathcal{C} \log m \left(\sqrt{m}\right)^p E_M(\mathbf{F}(f))_u.$$

Since $f \in W_{p+r+1}(u)$, by [18, Lemma 5.4, p. 5674] it follows $\mathbf{F}(f) \in W_{p+r}(u)$ and, using [18, Lemma 5.5], we deduce

$$T_1 \leq \frac{\mathcal{C}}{(\sqrt{m})^r} \log m \|f\|_{W_{p+r+1}(u)}. \tag{46}$$

By the Bernstein inequality [3, (3.4)] and [11, Theorem 2.2]

$$T_2 \leq \mathcal{C}\sqrt{m^p}\|L_{m+1,1}(w, \mathbf{F}(f) - \mathbf{F}_m(f))u\| \leq \mathcal{C}\sqrt{m^p} \log m [E_M(\mathbf{F}(f))_u + E_M(\mathbf{F}_m(f))_u].$$

Using [18, Lemma 5.5] we deduce

$$T_2 \leq \mathcal{C}\frac{\log m}{(\sqrt{m})^r}\|f\|_{W_{p+r+1}(u)}.$$

(17) follows combining last estimate and (46) with (45).

We omit the proof of (16), since it similar to the proof of (13). □

Proof of Theorem 3.3. Start from

$$|\Psi_{p,m}(f,t)u(t)\varphi^p(t)| \leq \|\rho_{p,m}(f)u\varphi^p\| + \sum_{k=1}^{p} \binom{p}{k} \|(f - L_{m+1}(w,f))^{(k)}u\varphi^k\| \sup_{t>0} |\mathbf{H}_{p-k}(w,t)|\, \varphi^{p-k}(t). \tag{47}$$

Taking into account Theorems 3.1 and 2.2 and Lemma 6.1, the thesis follows. □

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