

On the numerical solution of a boundary integral equation for the exterior Neumann problem on domains with corners[☆]

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Abstract

The authors propose a “modified” Nyström method to approximate the solution of a boundary integral equation connected with the exterior Neumann problem for Laplace’s equation on planar domains with corners. They prove the convergence and the stability of the method and show some numerical tests.

Keywords: Boundary integral equations, Neumann problem, Nyström method

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1. Introduction

Let us consider the exterior Neumann problem for Laplace’s equation

$$\left\{ \begin{array}{ll} \Delta u(P) = 0, & P \in \mathbb{R}^2 \setminus \overline{D}, \\ \frac{\partial u(P)}{\partial n_P} = f(P), & P \in \Sigma, \\ |u(P)| = o(1), & |P| \rightarrow \infty, \end{array} \right. \quad (1)$$

[☆]Dedicated to Professor Giuseppe Mastroianni on the occasion of his 75th birthday

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where D is a bounded open simply connected region in the plane with the boundary Σ , n_P is the inward normal vector to Σ at P , and f is a given sufficiently smooth function on Σ satisfying

$$\int_{\Sigma} f(P) d\Sigma_P = 0.$$

The aim is to look for solutions of the problem (1) which are continuous on $\mathbb{R}^2 \setminus D$. Note that this problem is uniquely solvable (see, for instance, [19, Theorem 6.24],[5, p. 351],[10, p. 152]).

In order to solve (1) via boundary integral equations, one can use the Green representation formula for potential functions on exterior regions

$$\begin{aligned} u(A) &= u(\infty) - \frac{1}{2\pi} \int_{\Sigma} \frac{\partial u(Q)}{\partial n_Q} \log |A - Q| d\Sigma_Q \\ &+ \frac{1}{2\pi} \int_{\Sigma} u(Q) \frac{\partial}{\partial n_Q} \log |A - Q| d\Sigma_Q, \quad A \in \mathbb{R}^2 \setminus \overline{D}, \end{aligned} \quad (2)$$

getting, by standard arguments (see, for instance, [3, pp. 388-389]), the boundary integral equation of the second kind

$$-(2\pi - \Omega(P))u(P) + \int_{\Sigma} u(Q) \frac{\partial}{\partial n_Q} \log |P - Q| d\Sigma_Q = g(P), \quad P \in \Sigma \quad (3)$$

where $\Omega(P)$ denotes the interior angle to Σ at P and

$$g(P) = \int_{\Sigma} f(Q) \log |P - Q| d\Sigma_Q. \quad (4)$$

Defining the operator

$$(Ku)(P) = (-\pi + \Omega(P))u(P) + \int_{\Sigma} u(Q) \frac{\partial}{\partial \mathbf{n}_Q} [\log |P - Q|] d\Sigma_Q, \quad P \in \Sigma \quad (5)$$

which is a bounded map from $C(\Sigma)$ into $C(\Sigma)$ (see [3, p. 389]), one can rewrite equation (3) in the following more compact operator form

$$(-\pi + K)u = g. \quad (6)$$

This is a boundary integral equation of direct type that, differently from those of indirect type obtained using the potential theory approach, presents

the advantage of getting the solution on the boundary without any further calculations. On the other hand, it involves the integral term (4) which is difficult to handle numerically, because of the presence of the logarithmic kernel.

We suppose that the boundary Σ is twice continuously differentiable with the exception of corners at points P_1, \dots, P_n with interior angles

$$\alpha_k = (1 - \chi_k)\pi, \quad -1 < \chi_k < 1, \quad \chi_k \neq 0, \quad k = 1, \dots, n. \quad (7)$$

We remark that under this assumption, the operator K is not compact.

Several integral equation methods like collocation, Galerkin and Nyström methods, having the purpose of approximating the solution of the Dirichlet or the Neumann problem in planar domains with corners, are available in literature. Most of them are based on the representation of the solution u in the form of a single or double layer potential and on the resolution of the corresponding boundary integral equations defined on piecewise smooth curves.

A variety of these methods makes use of piecewise polynomial approximations on graded meshes [11, 18, 20, 26] which, on the one hand, allow to achieve arbitrarily high order of convergence, on the other hand, could produce ill-conditioned linear systems as the local degree increases.

Sometimes [20, 25] such approaches are combined with smoothing strategies whose purpose is to improve the rate of convergence of the proposed numerical procedure.

More recently, an extensive literature on efficient numerical methods to discretize boundary integral equations connected with elliptic problems on domains with corners has been developed (see [6, 7, 8, 9, 10, 16, 17] and the references therein).

For instance, in [10] a new algorithm for the solution of the Neumann problem for the Laplace equation is described. Since the solution of the corresponding boundary integral equation can be unbounded at the corners of the domain, the method proposes the analytical subtraction of singularities in order to get high accuracy and, also, a special treatment of nearly non-integrable integrands in such a way to avoid cancellation errors.

Methods of Nyström type based on discretization techniques [8, 9], as well as compression and preconditioning schemes for the arising linear systems are proposed in [6, 7]. Such procedures allow one to produce well-conditioned linear systems that do not become too large for domains with piecewise smooth

boundaries having a great number of corners. Unfortunately, stability and convergence results have not been proved theoretically but only supported by numerical evidence. In fact several numerical examples show that high accuracy is achieved in the computation of the solutions.

In this paper, we do not seek the solution in the form of a potential but we solve the boundary integral equation (3) computing directly the harmonic function u , at first on the boundary and then, by using (2), on the exterior domain. Let us note that the same equation with a different right-hand side arises for the indirect boundary integral equation applied to the Dirichlet problem of Laplace's equation over the interior domain D . Thus, for its numerical solution we follow the idea that we proposed in [14] by applying a "modified" Nyström-type method based on global approximation. However, in (3) the right-hand side does not coincide with the data on the boundary as in the Dirichlet problem, but is given by the integral (4) which involves the Neumann data f . This integral needs to be approximated if its analytical expression is not known. Moreover, once equation (3) is solved, in most cases a suitable approximation of the integrals appearing in (2) is also necessary in order to compute the numerical solution of the original problem (1).

In other words, in this paper we address the problem of the numerical treatment of the integrals appearing in (2) and (4) (taking also into account the presence of their logarithmic kernel) combined with the "modified" Nyström-type method that we described in [14]. In this way, we propose a numerical procedure for the numerical solution of the exterior Neumann problem for Laplace's equation on planar domains with piecewise smooth boundaries which, to our knowledge, has never been investigated before.

The first step of our method is the introduction of a suitable decomposition of the boundary in order to rewrite equation (3) as an equivalent system of integral equations. More precisely, we divide each smooth arc of the boundary in three sections, choosing the two-non central parts of a very small length such that they essentially coincide with the straight segments tangent to the curve at the corner points. Then, we compute its solution by applying a Nyström method based on global approximation on each section of the boundary (see [25], [14]). The method uses a Radau quadrature formula, based on different numbers of quadrature knots according to the different lengths of the smooth sections involved in the adopted decomposition of the boundary. Nevertheless, in order to be able to establish stability and convergence results, we need to modify slightly the discrete operator, approximating the operator K , around the corners. We remark that this

modification is not only theoretical but it is also performed numerically.

A complete analysis of the convergence and the stability of the proposed procedure is conducted, by showing that the method can be applied to any domain D , regardless of the combination of interior angles. Moreover, error estimates are provided and it is also proved that the method always leads to solve well-conditioned linear systems.

The paper is structured as follows. In Section 2 we introduce some functional spaces and quadrature formulas. In Section 3 we present the method by giving convergence and stability results. Section 4 is dedicated to the proofs. Finally, in Section 5 we show some numerical tests.

2. Preliminaries

2.1. Spaces of functions

Let us denote by w a weight function on $[0, 1]$ and define the space L_w^p , $1 \leq p < \infty$ as the set of all measurable functions such that

$$\|fw\|_p = \left(\int_0^1 |f(x)w(x)|^p dx \right)^{\frac{1}{p}}, \quad 1 \leq p < +\infty.$$

Let us also introduce the Sobolev-type subspace $W_r^p(w)$ of $L_w^p([0, 1])$ defined as follows

$$W_r^p(w) = \{f \in L_w^p([0, 1]) \mid f^{(r-1)} \in AC((0, 1)), \|f^{(r)}\varphi^r w\|_p < \infty\},$$

where r is a positive integer, $\varphi(x) = \sqrt{x(1-x)}$, and $AC((0, 1))$ denotes the collection of all functions which are absolutely continuous on every closed subset of $(0, 1)$. We equip this space with the norm

$$\|f\|_{W_r^p(w)} = \|fw\|_p + \|f^{(r)}\varphi^r w\|_p.$$

If $w \equiv 1$ we simply write W_r^p instead of $W_r^p(w)$.

Finally, as usual, for $r \in \mathbb{N}_0 \cup \{\infty\}$ we denote by $C^r([0, 1])$ the set of all continuous functions with r continuous derivatives and we introduce the product space

$$C^r([0, 1])^m = \underbrace{C^r([0, 1]) \times C^r([0, 1]) \times \dots \times C^r([0, 1])}_{m \text{ times}}.$$

For $r = 0$, the space $C^0([0, 1])^m = C([0, 1])^m$ equipped with the norm

$$\|(f_1, f_2, \dots, f_m)\|_\infty = \max_{i=1,2,\dots,m} \|f_i\|_\infty \quad (8)$$

is complete.

2.2. Quadrature rules

In this subsection we report the quadrature formulas we adopt in the numerical method and we mention some results which will be useful in the sequel.

Denoting by $\{p_m(v^{\alpha,\beta})\}_m$ the sequence of polynomials which are orthonormal on $[0, 1]$ with respect to the Jacobi weight $v^{\alpha,\beta}(x) = x^\alpha(1-x)^\beta$, let $x_{m,k}^{\alpha,\beta}$, $k = 1, \dots, m$, be the zeros of $p_m(v^{\alpha,\beta})$ and $l_k^{\alpha,\beta}$, $k = 1, \dots, m$, be the fundamental Lagrange polynomials based on these points. Then, according to this notation, the Gauss-Legendre quadrature formula [13] reads as

$$\int_0^1 f(x)dx = \sum_{k=1}^m \lambda_{m,k}^L f(x_{m,k}^L) + e_m^L(f), \quad (9)$$

where $x_{m,k}^L = x_{m,k}^{0,0}$, $\lambda_{m,k}^L = \int_0^1 l_k^{0,0}(x)dx$, $\forall k \in \{1, \dots, m\}$, and e_m^L is the remainder term, while the Gauss-Radau formula [13] is given by

$$\int_0^1 f(x)dx = \sum_{k=0}^m \lambda_{m,k}^R f(x_{m,k}^R) + e_m^R(f), \quad (10)$$

with $x_{m,0}^R = 0$, $\lambda_{m,0}^R = \frac{1}{(m+1)^2}$, $x_{m,k}^R = x_{m,k}^{0,1}$, $\lambda_{m,k}^R = \int_0^1 l_k^{0,1}(x)v^{0,1}(x)dx$, $\forall k \in \{1, \dots, m\}$, and e_m^R the quadrature error.

In the next theorem we give an estimate for $e_m^L(f)$ and $e_m^R(f)$. To this end, we recall the definition of the weighted error of best polynomial approximation

$$E_m(f)_{w,p} = \inf_{P_m \in \mathbb{P}_m} \|(f - P_m)w\|_p,$$

where \mathbb{P}_m is the set of all algebraic polynomials of degree at most m .

Moreover, in the following, \mathcal{C} denotes a positive constant which may assume different values in different formulas and we shall write $\mathcal{C} \neq \mathcal{C}(a, b, \dots)$ to say that \mathcal{C} is independent of the parameters a, b, \dots .

The next theorem follows from [21, Theorem 5.1.8, p. 338] and the Favard inequality (see [21, p. 171]).

Theorem 2.1. *For all $f \in W_r^1$, $r \geq 1$, it results*

$$|e_m^L(f)| \leq \frac{\mathcal{C}}{m^r} E_{2m-1-r}(f^{(r)})_{\varphi^r,1} \quad (11)$$

and

$$|e_m^R(f)| \leq \frac{\mathcal{C}}{m^r} E_{2m-r} (f^{(r)})_{\varphi^r, 1}, \quad (12)$$

where $\varphi(x) = \sqrt{x(1-x)}$ and $\mathcal{C} \neq \mathcal{C}(m, f)$.

3. The method

In this section we describe the numerical method we propose in order to approximate the solution of problem (1). The procedure consists in three steps. The first one is to rewrite equation (3) as an equivalent system of integral equations by using a suitable decomposition of the boundary. The second step is to solve this system by applying a Nyström-type method based on the Gauss-Radau formula (10), with a number of quadrature knots depending on the length of the involved arc of the boundary. Finally, the solution (2) of the exterior Neumann problem is approximated using the results obtained in the previous step.

3.1. An equivalent system of integral equations

By proceeding in counterclockwise direction, we denote by Γ_k and Υ_k two sufficiently small smooth arcs of the boundary Σ intersecting at the corner P_k . Moreover, let τ_k and t_k be the segments tangent to the arcs Γ_k and Υ_k at P_k , respectively. We assume that τ_k has the same length of Γ_k and t_k has the same length of Υ_k . We also suppose that

$$\forall P \in \Gamma_k \setminus \{P_k\} : \min_{P' \in \tau_k} |P - P'| \leq \delta |P - P_k| \quad (13)$$

and

$$\forall Q \in \Upsilon_k \setminus \{P_k\} : \min_{Q' \in t_k} |Q - Q'| \leq \delta |Q - P_k| \quad (14)$$

with δ a very small positive number. Such hypothesis will allow us to represent (see (22)), in a neighborhood of the corner points P_k , the involved integral operator as the sum of a compact operator and a bounded operator (see [4],[12]).

Then, denoting by C_k the section connecting Υ_k and Γ_{k+1} , with $\Gamma_{n+1} \equiv \Gamma_1$, and collectively by $\Sigma_1, \dots, \Sigma_{3n}$ all these sections, starting from Γ_1 , we have

$$\Sigma = \bigcup_{j=1}^{3n} \Sigma_j, \quad \text{with} \quad \Sigma_j = \begin{cases} \Gamma_k, & j \equiv 1 \pmod{3}, & k = \frac{j-1}{3} + 1 \\ \Upsilon_k, & j \equiv 2 \pmod{3}, & k = \frac{j-2}{3} + 1 \\ C_k, & j \equiv 0 \pmod{3}, & k = \frac{j-3}{3} + 1 \end{cases}. \quad (15)$$

Moreover, setting $Q_j = \Sigma_j \cap \Sigma_{j+1}$, ($\Sigma_{3n+1} \equiv \Sigma_1$) we define the subspace \mathcal{S} of $C(\Sigma_1) \times \dots \times C(\Sigma_{3n})$ as

$$\mathcal{S} = \{(f_1, \dots, f_{3n}) \mid f_j \in C(\Sigma_j), f_j(Q_j) = f_{j+1}(Q_j), \forall j \in \{1, \dots, 3n\}\}.$$

Now, we consider the following system of $3n$ boundary integral equations

$$(-2\pi + \Omega(P))u_i(P) + \sum_{j=1}^{3n} \int_{\Sigma_j} u_j(Q) \frac{\partial}{\partial \mathbf{n}_Q} [\log |P - Q|] d\Sigma_Q = g_i(P), \quad (16)$$

$$P \in \Sigma_i, \quad i = 1, \dots, 3n,$$

where u_i and g_i denote the restrictions of the functions u and g to the curve Σ_i , respectively. Let us observe that from each solution $u \in C(\Sigma)$ of (3) one gets a solution $(u_1, \dots, u_{3n}) \in \mathcal{S}$ of (16) and vice versa.

In order to transform the above curvilinear $2D$ integrals into $1D$ integrals, for each arc Σ_i we introduce a parametric representation σ_i defined on the interval $[0, 1]$

$$\sigma_i : s \in [0, 1] \rightarrow (\xi_i(s), \eta_i(s)) \in \Sigma_i, \quad (17)$$

with $\sigma_i \in C^2([0, 1])$ and $|\sigma'_i(s)| \neq 0$ for each $0 \leq s \leq 1$ and $i = 1, \dots, 3n$. Moreover, without any loss of generality, we assume that the parametrizations σ_i are chosen such that $\sigma_i(0) = P_k$ if $\Sigma_i = \Gamma_k$ or $\Sigma_i = \Upsilon_k$, with σ_i traversing $\Sigma_i = \Gamma_k$ in a clockwise direction and σ_i traversing $\Sigma_i = \Upsilon_k$ and $\Sigma_i = C_k$ in a counter-clockwise direction. Hence, system (16) becomes, for $s \in [0, 1]$,

$$(-2\pi + \bar{\Omega}_i(s))\bar{u}_i(s) + \sum_{j=1}^{3n} \int_0^1 K^{i,j}(t, s) \bar{u}_j(t) dt = \bar{g}_i(s), \quad i = 1, \dots, 3n, \quad (18)$$

where $\bar{\Omega}_i(s) = \Omega(\sigma_i(s))$, $\bar{u}_i(s) = u_i(\sigma_i(s))$, $\bar{g}_i(s) = g_i(\sigma_i(s))$ and

$$K^{i,j}(t, s) = \begin{cases} a \frac{\eta'_j(t)[\xi_i(s) - \xi_j(t)] - \xi'_j(t)[\eta_i(s) - \eta_j(t)]}{[\xi_i(s) - \xi_j(t)]^2 + [\eta_i(s) - \eta_j(t)]^2}, & i \neq j \quad \text{or} \quad t \neq s \\ a \frac{1}{2} \frac{\eta'_j(t)\xi''_j(t) - \xi'_j(t)\eta''_j(t)}{[\xi'_j(t)]^2 + [\eta'_j(t)]^2}, & i = j \quad \text{and} \quad t = s \end{cases}, \quad (19)$$

for $t, s \in [0, 1]$ with $a = -1$ if $j \equiv 1 \pmod{3}$ and $a = 1$ if $j \equiv 2, 0 \pmod{3}$. We point out that, in virtue of our assumptions,

$$\bar{\Omega}_i(s) = \bar{\Omega}_{i+1}(s) = \begin{cases} \pi, & 0 < s \leq 1 \\ (1 - \chi_k)\pi, & s = 0 \end{cases} \quad (20)$$

when $i \equiv 1 \pmod{3}$ and $k = \frac{i-1}{3} + 1$, and $\bar{\Omega}_i(s) = \pi, \forall s \in [0, 1]$, if $i \equiv 0 \pmod{3}$.

In order to carry out the numerical treatment of system (18), we introduce the operators

$$(\mathcal{K}^{i,j}\rho)(s) = \int_0^1 K^{i,j}(t, s)\rho(t)dt, \quad \rho \in C([0, 1]), \quad (21)$$

which are compact on the space $C([0, 1])$, since their kernels are continuous on $[0, 1] \times [0, 1]$ except when $i, j \equiv 1, 2 \pmod{3}$ and $|i - j| = 1$ (see, for instance, [3, p. 7], [20, p. 158]). In fact, in such cases $\mathcal{K}^{i,j}$ takes the following form (see [4, (7.16)], [12, p. 127], [18, p. 238])

$$(\mathcal{K}^{i,j}\rho)(s) = (\mathcal{L}^{i,j}\rho)(s) + (\mathcal{M}^{i,j}\rho)(s), \quad (22)$$

where the integral operators $\mathcal{L}^{i,j}$

$$(\mathcal{L}^{i,j}\rho)(s) = \int_0^1 L^{i,j}(t, s)\rho(t)dt, \quad (23)$$

have a Mellin-type kernel given by

$$L^{i,j}(t, s) = -\frac{s \sin(\chi_k \pi)}{s^2 + 2ts \cos(\chi_k \pi) + t^2},$$

with $k = \frac{i-1}{3} + 1$ if $i \equiv 1 \pmod{3}$, and $k = \frac{i-2}{3} + 1$ if $i \equiv 2 \pmod{3}$, while the integral operators $\mathcal{M}^{i,j}$

$$(\mathcal{M}^{i,j}\rho)(s) = \int_0^1 M^{i,j}(t, s)\rho(t)dt, \quad \rho \in C([0, 1]), \quad (24)$$

have the kernel

$$M^{i,j}(t, s) = \begin{cases} K^{i,j}(t, s) - L^{i,j}(t, s) & (t, s) \neq (0, 0) \\ 0 & (t, s) = (0, 0) \end{cases},$$

continuous except possibly when $t = s = 0$ ([4]).

Thus, by collecting the Mellin-type integral operators in the following block matrix

$$\mathcal{L} = \begin{pmatrix} \mathcal{L}^1 & & & \\ & \mathcal{L}^4 & & \\ & & \ddots & \\ & & & \mathcal{L}^{3n-2} \end{pmatrix}, \quad (25)$$

with the blocks \mathcal{L}^i , for $i = 1, \dots, 3n$, and $i \equiv 1 \pmod{3}$, given by

$$\mathcal{L}^i = \begin{pmatrix} (-\pi + \bar{\Omega}_i)I & \mathcal{L}^{i,i+1} & 0 \\ \mathcal{L}^{i+1,i} & (-\pi + \bar{\Omega}_{i+1})I & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad (26)$$

and all the remainder integral operators in the matrix

$$\mathcal{K} = \begin{pmatrix} \mathcal{K}^{1,1} & \mathcal{M}^{1,2} & \mathcal{K}^{1,3} & \dots & \mathcal{K}^{1,3n-2} & \mathcal{K}^{1,3n-1} & \mathcal{K}^{1,3n} \\ \mathcal{M}^{2,1} & \mathcal{K}^{2,2} & \mathcal{K}^{2,3} & \dots & \mathcal{K}^{2,3n-2} & \mathcal{K}^{2,3n-1} & \mathcal{K}^{2,3n} \\ \mathcal{K}^{3,1} & \mathcal{K}^{3,2} & \mathcal{K}^{3,3} & \ddots & \mathcal{K}^{3,3n-2} & \mathcal{K}^{3,3n-1} & \mathcal{K}^{3,3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ \mathcal{K}^{3n-2,1} & \mathcal{K}^{3n-2,2} & \mathcal{K}^{3n-2,3} & \dots & \mathcal{K}^{3n-2,3n-2} & \mathcal{M}^{3n-2,3n-1} & \mathcal{K}^{3n-2,3n} \\ \mathcal{K}^{3n-1,1} & \mathcal{K}^{3n-1,2} & \mathcal{K}^{3n-1,3} & \dots & \mathcal{M}^{3n-1,3n-2} & \mathcal{K}^{3n-1,3n-1} & \mathcal{K}^{3n-1,3n} \\ \mathcal{K}^{3n,1} & \mathcal{K}^{3n,2} & \mathcal{K}^{3n,3} & \dots & \mathcal{K}^{3n,3n-2} & \mathcal{K}^{3n,3n-1} & \mathcal{K}^{3n,3n} \end{pmatrix} \quad (27)$$

we can rewrite system (18), in a compact form, as

$$(-\pi\mathcal{I} + \mathcal{L} + \mathcal{K})\bar{u} = \bar{g}, \quad (28)$$

where

$$\mathcal{I} = \begin{pmatrix} I & 0 & \dots & 0 \\ 0 & I & \dots & 0 \\ \vdots & \dots & \ddots & \vdots \\ 0 & \dots & \dots & I \end{pmatrix}, \quad (29)$$

with I the identity operator on $C([0, 1])$, and

$$\bar{u} = (\bar{u}_1, \bar{u}_2, \dots, \bar{u}_{3n})^T, \quad \bar{g} = (\bar{g}_1, \bar{g}_2, \dots, \bar{g}_{3n})^T. \quad (30)$$

Remark

Let us note that in the case when the boundary Σ is a polygonal one, for each $i \equiv 1, 2 \pmod{3}$ we have that $\mathcal{K}^{i,j} \equiv \mathcal{L}^{i,j}$ if $|i - j| = 1$ and $\mathcal{K}^{i,j} \equiv 0$ if $i = j$.

Now, let us introduce the following complete subspace of $C([0, 1])^{3n}$

$$\begin{aligned} \tilde{\mathcal{X}} = \Big\{ (f_1, \dots, f_{3n})^T \in C([0, 1])^{3n} \mid & f_i(0) = f_{i+1}(0), f_{i+1}(1) = f_{i+2}(0), \\ & f_i(1) = f_{i-1}(1), \forall i \in \{1, \dots, 3n\}, i \equiv 1 \pmod{3} \Big\}, \end{aligned} \quad (31)$$

with $f_0 \equiv f_{3n}$, and the bijective map $\eta : C(\Sigma) \rightarrow \tilde{\mathcal{X}}$ defined as

$$\eta f = (\bar{f}_1, \dots, \bar{f}_{3n}), \quad \bar{f}_i(t) = f(\sigma_i(t)), \quad t \in [0, 1].$$

We note that the arrays \bar{u} and \bar{g} introduced in (30) belong to $\tilde{\mathcal{X}}$. Moreover, the operator $-\pi + K : C(\Sigma) \rightarrow C(\Sigma)$ being bijective [19, p. 78], the operator

$$(-\pi\mathcal{I} + \mathcal{L} + \mathcal{K})^{-1} : \tilde{\mathcal{X}} \rightarrow \tilde{\mathcal{X}}$$

exists and is bounded since we can write

$$(-\pi\mathcal{I} + \mathcal{L} + \mathcal{K}) = \eta(-\pi + K)\eta^{-1}. \quad (32)$$

However, in order to carry out the analysis of the stability and the convergence of the numerical procedure we are going to propose for approximating the solution of (28), let us also introduce the following complete subspace of $C([0, 1])^{3n}$

$$\mathcal{X} = \Big\{ (f_1, \dots, f_{3n})^T \in C([0, 1])^{3n} \mid f_i(0) = f_{i+1}(0), \forall i \in \{1, \dots, 3n\}, i \equiv 1 \pmod{3} \Big\} \quad (33)$$

equipped with the uniform norm defined in (8). Let us observe that $\tilde{\mathcal{X}} \subset \mathcal{X}$.

The next result holds true.

Theorem 3.1. *The system (28) has a unique solution \bar{u} in \mathcal{X} for each given right-hand side $\bar{g} \in \mathcal{X}$. Moreover, if $\bar{g} \in \tilde{\mathcal{X}}$ then $\bar{u} \in \tilde{\mathcal{X}}$.*

We remark that the solution \bar{u} has a low smoothness near the corner points. In fact, if we look in detail the smoothness properties of the solution on each section Σ_i of the boundary, it results that (see [3, 11, 15] and the references therein)

- for $i \equiv 0 \pmod{3}$, \bar{u}_i is smooth;
- for $i \equiv 1 \pmod{3}$, with $k = \frac{i-1}{3} + 1$, or $i \equiv 2 \pmod{3}$, with $k = \frac{i-2}{3} + 1$, we have

$$\bar{u}_i(t) = O(t^{\beta_k}), \quad 0 < t \leq 1, \quad \beta_k = \frac{1}{1 + |\chi_k|}, \quad (34)$$

$$\bar{u}_i^{(r)}(t) \leq \mathcal{C}t^{\beta_k - r}, \quad 0 < t \leq 1, \quad r = 1, 2, \dots \quad (35)$$

Note that, being $\frac{1}{2} < \beta_k < 1$, the first derivative of the solution has an algebraic singularity in the corner points P_k .

3.2. A Nyström method

In order to approximate the solution \bar{u} of (28), we introduce the finite rank operators defined as follows. When $j \equiv 0 \pmod{3}$, for each i , let

$$(\mathcal{K}_m^{i,j}\rho)(s) = \sum_{h=0}^{\nu_m} \lambda_{\nu_m,h}^R K^{i,j}(x_{\nu_m,h}^R, s) \rho(x_{\nu_m,h}^R) \quad (36)$$

be the discrete operator, approximating $\mathcal{K}^{i,j}$ in (21), obtained by applying the Radau formula (10) with $\nu_m = \nu(m)$ quadrature points. In the remainder cases that is for $j \equiv 1, 2 \pmod{3}$, let us define the operators

$$(\mathcal{K}_m^{i,j}\rho)(s) = \sum_{h=0}^{\mu_m} \lambda_{\mu_m,h}^R K^{i,j}(x_{\mu_m,h}^R, s) \rho(x_{\mu_m,h}^R), \quad (37)$$

for any i such that $|i - j| \neq 1$ and

$$(\mathcal{M}_m^{i,j}\rho)(s) = \sum_{h=0}^{\mu_m} \lambda_{\mu_m,h}^R M^{i,j}(x_{\mu_m,h}^R, s) \rho(x_{\mu_m,h}^R), \quad (38)$$

$$(\mathcal{L}_m^{i,j}\rho)(s) = \sum_{h=0}^{\mu_m} \lambda_{\mu_m,h}^R L^{i,j}(x_{\mu_m,h}^R, s) \rho(x_{\mu_m,h}^R), \quad (39)$$

if $|i - j| = 1$. Such operators approximate $\mathcal{K}^{i,j}$, $\mathcal{M}^{i,j}$ and $\mathcal{L}^{i,j}$ defined in (21), (24) and (23), respectively, by means of the same quadrature rule with

$\mu_m = \mu(m)$ nodes.

From now on we shall assume $\nu_m := m$ and $\mu_m := [qm]$, with $q < 1$.

At this point, if we apply the Nyström method based on these quadrature formulas, we should solve the following approximating system

$$(-\pi\mathcal{I} + \mathcal{L}_m + \mathcal{K}_m)\bar{u}_m = \bar{g}, \quad (40)$$

where \mathcal{L}_m is the matrix obtained by replacing in (25) the blocks \mathcal{L}^i with

$$\mathcal{L}_m^i = \begin{pmatrix} (-\pi + \bar{\Omega}_i)I & \mathcal{L}_m^{i,i+1} & 0 \\ \mathcal{L}_m^{i+1,i} & (-\pi + \bar{\Omega}_{i+1})I & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad (41)$$

\mathcal{K}_m is the matrix given by (27) but with the operators $\mathcal{K}_m^{i,j}$ and $\mathcal{M}_m^{i,j}$ in place of $\mathcal{K}^{i,j}$ and $\mathcal{M}^{i,j}$, respectively, and $\bar{u}_m = (\bar{u}_{m,1}, \dots, \bar{u}_{m,3n})^T$ is the array of the unknowns. However, proceeding in this way we will not be able to prove the stability and convergence of the method. Indeed, it is possible to establish (see Theorem 3.3) that any sequence of operators $\{\mathcal{K}_m^{i,j}\}_m$ and $\{\mathcal{M}_m^{i,j}\}_m$ are pointwise convergent to the operators $\mathcal{K}^{i,j}$ and $\mathcal{M}^{i,j}$, respectively in the space $C([0, 1])$, whereas we can not state a similar result for the sequences $\{\mathcal{L}_m^{i,j}\}_m$. More precisely, we are able to prove that, for any $\rho \in C([0, 1])$, each sequence of functions $\{\mathcal{L}_m^{i,j}\rho\}_m$ converges uniformly to $\mathcal{L}^{i,j}\rho$ in any interval of the type $[\frac{c}{m^{2-2\epsilon}}, 1]$, for some constant $c > 0$ and arbitrarily small $\epsilon > 0$, but not in the whole interval $[0, 1]$.

To overcome this problem, we propose a perturbed Nyström method based on a modification of the matrix \mathcal{L}_m . Indeed, following an idea in [14], we modify the matrix \mathcal{L}_m by replacing the blocks \mathcal{L}_m^i with the new blocks $\tilde{\mathcal{L}}_m^i$ defined as

$$\begin{aligned} & (\tilde{\mathcal{L}}_m^i \varrho)(s) \\ &= \begin{cases} (\mathcal{L}_m^i \varrho)(s), & \frac{c}{m^{2-2\epsilon}} \leq s \leq 1, \\ \frac{m^{2-2\epsilon}}{c} \left[s(\mathcal{L}_m^i \varrho) \left(\frac{c}{m^{2-2\epsilon}} \right) + \left(\frac{c}{m^{2-2\epsilon}} - s \right) (\mathcal{L}_m^i \varrho)(0) \right], & 0 \leq s < \frac{c}{m^{2-2\epsilon}}, \end{cases} \end{aligned} \quad (42)$$

for $i = 1, \dots, 3n$, $i \equiv 1 \pmod{3}$, $\varrho \in C([0, 1])$, where $c > 0$ is a fixed constant and $\epsilon > 0$ is an arbitrarily small number.

Then, denoting by $\tilde{\mathcal{L}}_m$ the matrix thus obtained, in place of (28), we consider the new approximating system

$$(-\pi\mathcal{I} + \tilde{\mathcal{L}}_m + \mathcal{K}_m)\bar{u}_m = \bar{g}. \quad (43)$$

The operators $\tilde{\mathcal{L}}_m$ and \mathcal{K}_m satisfy the following theorems.

Theorem 3.2. *The operators $\tilde{\mathcal{L}}_m : \mathcal{X} \rightarrow \mathcal{X}$ are linear maps such that*

$$\limsup_{m \rightarrow \infty} \|\tilde{\mathcal{L}}_m\| < \pi \quad (44)$$

and

$$\lim_{m \rightarrow \infty} \|(\tilde{\mathcal{L}}_m - \mathcal{L})\rho\|_\infty = 0, \quad \forall \rho \in \mathcal{X}. \quad (45)$$

Theorem 3.3. *The operators $\mathcal{K}_m : \mathcal{X} \rightarrow \mathcal{X}$ are linear maps such that the set $\{\mathcal{K}_m\}_m$ is collectively compact and*

$$\lim_{m \rightarrow \infty} \|(\mathcal{K}_m - \mathcal{K})\rho\|_\infty = 0, \quad \forall \rho \in \mathcal{X}. \quad (46)$$

Now, in order to compute the solution of the approximating system (43), we collocate suitably the equations of (43) in the quadrature nodes, getting an equivalent linear system. More precisely, if we denote by $\bar{\psi}_m = (\bar{\psi}_{m,1}, \bar{\psi}_{m,2}, \dots, \bar{\psi}_{m,3n})$ the array

$$\bar{\psi}_m = (-\pi\mathcal{I} + \tilde{\mathcal{L}}_m + \mathcal{K}_m)\bar{u}_m,$$

the linear system consists of the following $M_m = n(2\mu_m + \nu_m + 3)$ equations

$$\begin{cases} \bar{\psi}_{m,i}(x_{\mu_m,\ell}^R) &= \bar{g}_i(x_{\mu_m,\ell}^R), & \ell = 0, \dots, \mu_m & \text{if } i \equiv 1, 2 \pmod{3} \\ \bar{\psi}_{m,i}(x_{\nu_m,\ell}^R) &= \bar{g}_i(x_{\nu_m,\ell}^R), & \ell = 0, \dots, \nu_m & \text{if } i \equiv 0 \pmod{3} \end{cases} \quad (47)$$

for $i = 1, \dots, 3n$, in the M_m unknowns

$$a_{i,\ell} = \begin{cases} \bar{u}_{m,i}(x_{\mu_m,\ell}^R), & \ell = 0, \dots, \mu_m & \text{if } i \equiv 1, 2 \pmod{3} \\ \bar{u}_{m,i}(x_{\nu_m,\ell}^R), & \ell = 0, \dots, \nu_m & \text{if } i \equiv 0 \pmod{3} \end{cases}. \quad (48)$$

Denoting by \mathbf{a} the array of the unknowns, by A_m the matrix of the coefficients, and by \mathbf{b} the right-hand side vector, we can rewrite the system (47) in the compact form

$$A_m \mathbf{a} = \mathbf{b}. \quad (49)$$

We remark that it is equivalent, in some sense, to the approximating problem (43). Indeed, let us denote by $\tilde{\mathbb{R}}^{M_m}$ the subspace of \mathbb{R}^{M_m} containing all the arrays

$$\mathbf{c} = (c_{1,0}, \dots, c_{1,\mu_m}, c_{2,0}, \dots, c_{2,\mu_m}, c_{3,0}, \dots, c_{3,\nu_m}, \dots, \\ c_{3n-2,0}, \dots, c_{3n-2,\mu_m}, \dots, c_{3n-1,0}, \dots, c_{3n-1,\mu_m}, \dots, c_{3n,0}, \dots, c_{3n,\nu_m})$$

such that $c_{i,0} = c_{i+1,0}$, $\forall i \in \{1, \dots, 3n\}$, $i \equiv 1 \pmod{3}$. Then, each solution $\bar{u}_m \in \mathcal{X}$ of (43) furnishes a solution \mathbf{a} of system (49) belonging to $\tilde{\mathbb{R}}^{M_m}$. It will be sufficient to evaluate the components of the vector \bar{u}_m at the suitable quadrature nodes of the Radau formula. Vice versa, if

$$\mathbf{a} = (a_{1,0}, \dots, a_{1,\mu_m}, a_{2,0}, \dots, a_{2,\mu_m}, a_{3,0}, \dots, a_{3,\nu_m}, \dots, \\ a_{3n-2,0}, \dots, a_{3n-2,\mu_m}, \dots, a_{3n-1,0}, \dots, a_{3n-1,\mu_m}, \dots, a_{3n,0}, \dots, a_{3n,\nu_m}) \in \tilde{\mathbb{R}}^{M_m}$$

satisfies (49), then there is a unique $\bar{u}_m \in \mathcal{X}$, solution of (43), such that the equalities (48) hold true.

Consequently, we can conclude that the operator $-\pi\mathcal{I} + \tilde{\mathcal{L}}_m + \mathcal{K}_m$ is invertible on the space \mathcal{X} if and only if the matrix A_m is invertible on $\tilde{\mathbb{R}}^{M_m}$.

The next theorem contains our main result.

Theorem 3.4. *Let Σ be a piecewise smooth curve in the plane which is the union of closed C^2 arcs whose end points P_1, \dots, P_n are the corners of Σ . Let $(1 - \chi_k)\pi$, $-1 < \chi_k < 1$, $\chi_k \neq 0$ be the interior angle at the corner point P_k , for each $k = 1, \dots, n$. Then, for sufficiently large m , say $m \geq m_0$, the operators $-\pi\mathcal{I} + \tilde{\mathcal{L}}_m + \mathcal{K}_m$ are invertible and their inverses are uniformly bounded on \mathcal{X} .*

Moreover, denoting by $\text{cond}(-\pi\mathcal{I} + \tilde{\mathcal{L}}_m + \mathcal{K}_m)$ the condition number of the operator $-\pi\mathcal{I} + \tilde{\mathcal{L}}_m + \mathcal{K}_m : \mathcal{X} \rightarrow \mathcal{X}$ and by $\text{cond}(A_m)$ the condition number of the matrix A_m in infinity norm, we have that, for any $m \geq m_0$,

$$\text{cond}(A_m) \leq \text{cond}(-\pi\mathcal{I} + \tilde{\mathcal{L}}_m + \mathcal{K}_m) \leq \mathcal{C} \quad (50)$$

where $\mathcal{C} \neq \mathcal{C}(m)$.

Furthermore, assuming $f \in C^p([0, 1])$ with p large enough, the solutions \bar{u} and \bar{u}_m of systems (28) and (43), respectively, satisfy the following error estimate

$$\|(\bar{u} - \bar{u}_m)(s)\|_\infty \leq \mathcal{C}[\|(\tilde{\mathcal{L}}_m - \mathcal{L})\bar{u}(s)\|_\infty + \|(\mathcal{K}_m - \mathcal{K})\bar{u}(s)\|_\infty] \quad (51)$$

where

$$\|(\tilde{\mathcal{L}}_m - \mathcal{L})\bar{u}(s)\|_\infty \leq \begin{cases} \mathcal{C} \max \left\{ \left(\frac{1}{m^{2-2\epsilon}} \right)^\beta, \frac{1}{m^{1+\epsilon}} \right\}, & s \in \left[0, \frac{c}{m^{2-2\epsilon}} \right] \\ \frac{\mathcal{C}}{m^2} \frac{1}{s^{\frac{1}{2}}}, & s \in \left[\frac{c}{m^{2-2\epsilon}}, 1 \right], \end{cases} \quad (52)$$

with ϵ as in (42), $\beta = \min_{k=1, \dots, n} \beta_k$, with β_k as in (34), and $\mathcal{C} \neq \mathcal{C}(m)$.

We remark that (see Theorem 3.3)

$$\lim_m \|(\mathcal{K}_m - \mathcal{K})\bar{u}(s)\|_\infty = 0, \quad \forall s \in [0, 1]$$

and the rate of convergence depends on the smoothness of the arcs of the boundary as well as on the behavior of the functions \bar{u}_j on the interval $[0, 1]$ (see (34), (35)).

Moreover, we note that the previous theorem establishes the convergence of the approximate solution \bar{u}_m to the exact one \bar{u} in the space \mathcal{X} (and not $\tilde{\mathcal{X}}$). Therefore, by reconstructing the approximate solution u_m on the initial boundary Σ , it can get a finite number of discontinuity points. Nevertheless, these discontinuities do not play any role when we replace the harmonic function u on Σ with u_m in (2) (see also (61)), in order to approximate the solution u of the Neumann problem at points of the exterior domain.

At this point, we investigate on a possible approximation of the right-hand side \bar{g} of system (43) in the case when it cannot be evaluated analytically.

To this end we decompose the whole boundary Σ in n smooth arcs $\tilde{\Sigma}_1, \dots, \tilde{\Sigma}_n$, with $\tilde{\Sigma}_k$ connecting the corner point P_k with P_{k+1} ($P_{n+1} \equiv P_1$) and represented by the parametrization $\tilde{\sigma}_k \in C^2([0, 1])$

$$\tilde{\sigma}_k : s \in [0, 1] \rightarrow (\tilde{\xi}_k(s), \tilde{\eta}_k(s)) \in \tilde{\Sigma}_k, \quad (53)$$

such that $|\tilde{\sigma}'_k(s)| \neq 0$ for each $0 \leq s \leq 1$. Then since, for any fixed $s \in [0, 1]$ and $i = 1, \dots, 3n$, we can write

$$\bar{g}_i(s) = g(\sigma_i(s)) = g(\tilde{\sigma}_\ell(s_i)), \quad (54)$$

for some $\ell = 1, \dots, n$ and with a suitable $s_i \in [0, 1]$, we focus our attention on the numerical computation of

$$g(\tilde{\sigma}_\ell(s)) = \sum_{k=1}^n \int_0^1 \phi_k(t) \log |\tilde{\sigma}_\ell(s) - \tilde{\sigma}_k(t)| dt, \quad s \in [0, 1] \quad (55)$$

where $\phi_k(t) = f(\tilde{\sigma}_k(t))|\tilde{\sigma}'_k(t)|$.

Now, if $\ell = k$ the computation of the logarithmic kernel, when t and s have a relative distance of the order of the machine precision eps , suffers from severe loss of accuracy, because of the numerical cancellation. Then, to avoid this situation we write

$$\log |\tilde{\sigma}_\ell(s) - \tilde{\sigma}_\ell(t)| = \log \frac{|\tilde{\sigma}_\ell(s) - \tilde{\sigma}_\ell(t)|}{|t - s|} + \log |t - s|$$

and if $|t - s| < eps$, for the first term at the right hand side, we use the approximation

$$\log \frac{|\tilde{\sigma}_\ell(s) - \tilde{\sigma}_\ell(t)|}{|t - s|} \simeq \log |\tilde{\sigma}'_\ell(t)|.$$

By this numerical trick, we can rewrite (55) as

$$\begin{aligned} g(\tilde{\sigma}_\ell(s)) &= \sum_{k=1, k \neq \ell}^n \int_0^1 \phi_k(t) \log |\tilde{\sigma}_\ell(s) - \tilde{\sigma}_k(t)| dt \\ &+ \int_0^1 \phi_\ell(t) [\log |t - s| + \delta_\ell(t, s)] dt \end{aligned} \quad (56)$$

where

$$\delta_\ell(t, s) = \begin{cases} \log |\tilde{\sigma}'_\ell(t)|, & |t - s| < eps, \\ \log \frac{|\tilde{\sigma}_\ell(s) - \tilde{\sigma}_\ell(t)|}{|t - s|}, & \text{otherwise} \end{cases}. \quad (57)$$

Now, in order to approximate the integrals appearing in (56), we propose to proceed as follows

- for $\ell \neq k$, we use the Gauss-Legendre quadrature formula (9) obtaining

$$\sum_{h=1}^M \lambda_{M,h}^L \phi_k(x_{M,h}^L) \log |\tilde{\sigma}_\ell(s) - \tilde{\sigma}_k(x_{M,h}^L)|,$$

- for $\ell = k$, we use a product integration rule for the first addendum getting

$$\sum_{h=1}^M \lambda_{M,h}^L \phi_\ell(x_{M,h}^L) \sum_{\nu=0}^{M-1} c_\nu(s) p_\nu(x_{M,h}^L),$$

with $p_\nu \equiv p_\nu(v^{0,0})$ the ν -th orthonormal Legendre polynomial and

$$c_\nu(s) = \int_0^1 p_\nu(z) \log |z - s| dz$$

the ν -th momentum computable by means of a recurrence formula (see, for instance, [22]) and we again adopt a Gauss-Legendre quadrature formula for the second term obtaining

$$\sum_{h=1}^M \lambda_{M,h}^L \phi_\ell(x_{M,h}^L) \delta_\ell(x_{M,h}^L, s).$$

Summarizing, we propose to approximate the right-hand side \bar{g} by

$$\bar{g}_M = (\bar{g}_{M,1}, \bar{g}_{M,2}, \dots, \bar{g}_{M,3n})$$

with

$$\begin{aligned} \bar{g}_{M,i}(s) &= \sum_{k=1, k \neq \ell}^n \sum_{h=1}^M \lambda_{M,h}^L \phi_k(x_{M,h}^L) \log |\tilde{\sigma}_\ell(s) - \tilde{\sigma}_k(x_{M,h}^L)| \\ &+ \sum_{h=1}^M \lambda_{M,h}^L \phi_\ell(x_{M,h}^L) \left(\sum_{\nu=0}^{M-1} c_\nu(s) p_\nu(x_{M,h}^L) + \delta_\ell(x_{M,h}^L, s) \right). \end{aligned} \quad (58)$$

The following theorem establishes the corresponding error estimate.

Theorem 3.5. *Let Σ be as in the assumption of Theorem 3.4 and $f \in C^p(\Sigma)$ with $p > 0$. Then, it results*

$$\|(\bar{g} - \bar{g}_M)(s)\|_\infty \leq \frac{\mathcal{C}}{M}, \quad \forall s \in [0, 1] \quad (59)$$

where $\mathcal{C} \neq \mathcal{C}(M)$.

Let us observe that under the hypothesis that the boundary $\Sigma \setminus \{P_1, \dots, P_n\}$ is $(q+2)$ -times differentiable, for some $q > 0$, the approximate right-hand side \bar{g}_M tends to the exact one \bar{g} with a rate of convergence of order $1/M^r$ where $r = \min\{q+1, p\}$.

We also remark that if we introduce this approximation of the right-hand side \bar{g} in the proposed numerical procedure, we really solve the following system

$$(-\pi \mathcal{I} + \tilde{\mathcal{L}}_m + \mathcal{K}_m) \bar{u}_m = \bar{g}_M \quad (60)$$

instead of (43). However in this case Theorem 3.4 still holds true but the error estimate becomes

$$\|(\bar{u} - \bar{u}_m)(s)\|_\infty \leq \mathcal{C}[\|(\tilde{\mathcal{L}}_m - \mathcal{L})\bar{u}(s)\|_\infty + \|(\mathcal{K}_m - \mathcal{K})\bar{u}(s)\|_\infty + \|(\bar{g} - \bar{g}_M)(s)\|_\infty].$$

3.3. Approximation of the Neumann solution

In this subsection we propose to approximate the solution of our initial problem (1), by taking advantage of the numerical results provided by the method described in the previous subsections.

To this end we note that, according to the parametric representation (53) of the arcs $\tilde{\Sigma}_k$ as well as to the decomposition (15) and the corresponding parametrizations (17), $\forall (x, y) \in \mathbb{R}^2 \setminus \bar{D}$ the solution $u(x, y)$, defined in (2), can be rewritten as

$$u(x, y) = -\frac{1}{2\pi} \left\{ \sum_{k=1}^n \int_0^1 \phi_k(t) \log |(\tilde{\xi}_k(t), \tilde{\eta}_k(t)) - (x, y)| dt - \sum_{i=1}^{3n} \int_0^1 H_i(x, y, t) \bar{u}_i(t) dt \right\}, \quad (61)$$

where, $\phi_k(t) = f(\tilde{\sigma}_k(t))|\tilde{\sigma}'_k(t)|$ for $k \in \{1, \dots, n\}$, $\bar{u}_i(t) = u_i(\sigma_i(t))$ for $i \in \{1, \dots, 3n\}$, and

$$H_i(x, y, t) = a \frac{\eta'_i(t)[x - \xi_i(t)] - \xi'_i(t)[y - \eta_i(t)]}{[x - \xi_i(t)]^2 + [y - \eta_i(t)]^2}$$

with $a = -1$ if $j \equiv 1 \pmod{3}$ and $a = 1$ if $j \equiv 2, 0 \pmod{3}$. We propose to approximate $u(x, y)$ by means of the function

$$u_m(x, y) = -\frac{1}{2\pi} \left\{ \sum_{k=1}^n \int_0^1 \phi_k(t) \log |(\tilde{\xi}_k(t), \tilde{\eta}_k(t)) - (x, y)| dt - \sum_{\substack{i=1 \\ i \equiv 1, 2 \pmod{3}}}^{3n} \sum_{k=0}^{\mu_m} \lambda_{\mu_m, k}^R H_i(x, y, x_{\mu_m, k}^R) \bar{u}_{m, i}(x_{\mu_m, k}^R) - \sum_{\substack{i=1 \\ i \equiv 0 \pmod{3}}}^{3n} \sum_{k=0}^{\nu_m} \lambda_{\nu_m, k}^R H_i(x, y, x_{\nu_m, k}^R) \bar{u}_{m, i}(x_{\nu_m, k}^R) \right\}, \quad (62)$$

obtained by replacing in (61) each \bar{u}_i with $\bar{u}_{m,i}$ and, then, by applying the suitable Radau quadrature formula in order to compute the integrals

$$\int_0^1 H_i(x, y, t) \bar{u}_{m,i}(t) dt.$$

Let us note that the quantities $\bar{u}_{m,i}(x_{\nu_m,k}^R)$ and $\bar{u}_{m,i}(x_{\mu_m,k}^R)$ involved in (62) are just the solutions of the linear system (47).

Moreover, when we are not able to compute analytically the integrals

$$\int_0^1 \phi_k(t) \log |(\tilde{\xi}_k(t), \tilde{\eta}_k(t)) - (x, y)| dt$$

in (62), we also approximate them by means of a suitable quadrature formula. For instance, we can use the Gauss-Legendre quadrature formula (9) with N nodes and, in this way, we get the approximate solution

$$\begin{aligned} u_{m,N}(x, y) = & -\frac{1}{2\pi} \left\{ \sum_{k=1}^n \sum_{h=1}^N \lambda_{N,h}^L \phi_k(x_{N,h}^L) \log |(\tilde{\xi}_k(x_{N,h}^L), \tilde{\eta}_k(x_{N,h}^L)) - (x, y)| \right. \\ & - \sum_{\substack{i=1 \\ i \equiv 1, 2 \pmod{3}}}^{3n} \sum_{k=0}^{\mu_m} \lambda_{\mu_m,k}^R H_i(x, y, x_{\mu_m,k}^R) \bar{u}_{m,i}(x_{\mu_m,k}^R) \\ & \left. - \sum_{\substack{i=1 \\ i \equiv 0 \pmod{3}}}^{3n} \sum_{k=0}^{\nu_m} \lambda_{\nu_m,k}^R H_i(x, y, x_{\nu_m,k}^R) \bar{u}_{m,i}(x_{\nu_m,k}^R) \right\}. \end{aligned} \quad (63)$$

The following theorem gives an error estimate for both the approximations (62) and (63).

Theorem 3.6. *Let the assumptions of Theorem 3.4 be satisfied and let u be the solution of the Neumann problem (1). Then, denoting by u_m and $u_{m,N}$ the approximations (62) and (63), respectively, $\forall (x, y) \in \mathbb{R}^2 \setminus \bar{D}$ we have the following pointwise error estimates*

$$|u(x, y) - u_m(x, y)| \leq \frac{\mathcal{C}}{m} \left(\frac{1}{d^2} + \frac{1}{d} \right) + \frac{\mathcal{C}'}{d} \|\bar{u} - \bar{u}_m\|_\infty, \quad (64)$$

$$|u(x, y) - u_{m,N}(x, y)| \leq \frac{\mathcal{C}}{m} \left(\frac{1}{d^2} + \frac{1}{d} \right) + \frac{\mathcal{C}'}{d} \|\bar{u} - \bar{u}_m\|_\infty + \frac{\mathcal{C}''}{dN}, \quad (65)$$

where $d = \min_{k=1,\dots,n} d_k$ with $d_k = \min_{0 \leq t \leq 1} |(x, y) - (\tilde{\xi}_k(t), \tilde{\eta}_k(t))|$, \mathcal{C} , \mathcal{C}' and \mathcal{C}'' are positive constants independent of (x, y) , m and N .

Let us remark that the first addendum on the right-hand side of (64) and (65) and the last addendum on the right-hand side of (65) could converge to zero with a rate greater than $1/m$ and $1/N$, respectively, if the boundary $\Sigma \setminus \{P_1, \dots, P_n\} \in C^{q+2}$, with $q > 0$. Moreover, from the previous estimates, we can deduce that the convergence becomes faster and faster the more the point (x, y) moves away from the boundary Σ .

4. Proofs

Proof of Theorem 3.1. From well known results (see, for instance, [3, p. 393]) it follows that for any array of functions $\rho = (\rho_1, \dots, \rho_{3n})^T \in \mathcal{X}$, setting $\varrho_i = (\rho_i, \rho_{i+1}, \rho_{i+2})^T \in C([0, 1])^3$, $\forall i \in \{1, \dots, 3n\}$, $i \equiv 1 \pmod{3}$, one has

$$\mathcal{L}^i \varrho_i \in C([0, 1])^3.$$

Moreover, it is easy to see that $\mathcal{L}\rho \in \mathcal{X}$ and that if $\|\rho\|_\infty \leq 1$, we have

$$\|\mathcal{L}\rho\|_\infty = \max_{\substack{i=1,\dots,3n \\ i \equiv 1 \pmod{3}}} \|\mathcal{L}^i \varrho_i\|_\infty \leq \max_{\substack{i=1,\dots,3n \\ i \equiv 1 \pmod{3}}} \left| \chi_{\frac{i-1}{3}+1} \right| \pi < \pi,$$

from which $\|\mathcal{L}\| < \pi$. Then, being $\mathcal{I} : \mathcal{X} \rightarrow \mathcal{X}$ with $\|-\pi\mathcal{I}\| = \pi$, by applying the geometric series theorem, we can deduce that the operator $(-\pi\mathcal{I} + \mathcal{L})^{-1} : \mathcal{X} \rightarrow \mathcal{X}$ exists and is bounded with

$$\|(-\pi\mathcal{I} + \mathcal{L})^{-1}\| \leq \frac{1}{\pi - \|\mathcal{L}\|}.$$

Consequently, equation (28) is equivalent to the following one

$$\bar{u} + (-\pi\mathcal{I} + \mathcal{L})^{-1} \mathcal{K} \bar{u} = (-\pi\mathcal{I} + \mathcal{L})^{-1} \bar{g}. \quad (66)$$

Now, let us note that the operator \mathcal{K} also maps \mathcal{X} into \mathcal{X} , since for any array of functions $\rho = (\rho_1, \dots, \rho_{3n})^T \in \mathcal{X}$, one has that $\mathcal{K}\rho \in C([0, 1])^{3n}$ ([4, p. 37]) and

$$(\mathcal{K}\rho)_i(0) = (\mathcal{K}\rho)_{i+1}(0) \quad \forall i \in \{1, \dots, 3n\}, \quad i \equiv 1 \pmod{3}.$$

In fact, for any fixed $i \in \{1, \dots, 3n\}$, $i \equiv 1 \pmod{3}$, we have

$$(\mathcal{K}^{i,j} \rho_j)(0) = (\mathcal{K}^{i+1,j} \rho_j)(0), \quad \forall j \in \{1, \dots, 3n\}, \quad j \neq i, i+1$$

and

$$\begin{aligned} (\mathcal{K}^{i,i} \rho_i)(0) + (\mathcal{M}^{i,i+1} \rho_{i+1})(0) &= \\ &= (\mathcal{K}^{i,i} \rho_i)(0) + (\mathcal{K}^{i,i+1} \rho_{i+1})(0) - (\mathcal{L}^{i,i+1} \rho_{i+1})(0) \\ &= (\mathcal{K}^{i+1,i} \rho_i)(0) + (\mathcal{K}^{i+1,i+1} \rho_{i+1})(0) - (\mathcal{L}^{i+1,i} \rho_i)(0) \\ &= (\mathcal{M}^{i+1,i} \rho_i)(0) + (\mathcal{K}^{i+1,i+1} \rho_{i+1})(0) \end{aligned}$$

being (see, for instance, [23, p. 274], [3, p. 393])

$$(\mathcal{L}^{i,i+1} \rho_{i+1})(0) = \rho_{i+1}(0) \chi_k \pi = \rho_i(0) \chi_k \pi = (\mathcal{L}^{i+1,i} \rho_i)(0)$$

with $k = \frac{i-1}{3} + 1$.

Moreover, \mathcal{K} is compact being matrix of compact operators (see [1, Theorem 3.2], [2, Proposition 4.4], [3, p. 7], [4, p. 37]). Consequently, the operator $(-\pi \mathcal{I} + \mathcal{L})^{-1} \mathcal{K} : \mathcal{X} \rightarrow \mathcal{X}$ is also compact. Thus for equation (28) the Fredholm alternative holds true and system (28) is unisolvent in \mathcal{X} , for each right-hand side $\bar{g} \in \mathcal{X}$, if and only if

$$\text{Ker}(-\pi \mathcal{I} + \mathcal{L} + \mathcal{K}) = \{0\}. \quad (67)$$

Now let us prove that (67) is fulfilled. Let $\bar{u} = (\bar{u}_1, \dots, \bar{u}_{3n})^T \in \text{Ker}(-\pi \mathcal{I} + \mathcal{L} + \mathcal{K})$. It is easy to check that $\bar{u} \in \tilde{\mathcal{X}}$. In fact from $(-\pi \mathcal{I} + \mathcal{L} + \mathcal{K})\bar{u} = 0$, it immediately follows that, for each $i \in \{1, 2, \dots, 3n\}$, $i \equiv 1 \pmod{3}$, $(\bar{u}_0 \equiv \bar{u}_{3n})$

$$-\pi \bar{u}_{i-1}(1) + \sum_{j=1}^{3n} (\mathcal{K}^{i-1,j} \bar{u}_j)(1) = 0$$

$$-\pi \bar{u}_i(1) + (\mathcal{L}^{i,i+1} \bar{u}_{i+1})(1) + \sum_{\substack{j=1 \\ j \neq i+1}}^{3n} (\mathcal{K}^{i,j} \bar{u}_j)(1) + (\mathcal{M}^{i,i+1} \bar{u}_{i+1})(1) = 0$$

and then, since one has that (see (17), (21), (22))

$$(\mathcal{K}^{i-1,j} \bar{u}_j)(1) = (\mathcal{K}^{i,j} \bar{u}_j)(1), \quad j \neq i+1$$

and

$$(\mathcal{K}^{i-1,i+1}\bar{u}_{i+1})(1) = (\mathcal{L}^{i,i+1}\bar{u}_{i+1})(1) + (\mathcal{M}^{i,i+1}\bar{u}_{i+1})(1),$$

we can deduce that $\bar{u}_{i-1}(1) = \bar{u}_i(1)$.

In a similar way, from $(-\pi\mathcal{I} + \mathcal{L} + \mathcal{K})\bar{u} = 0$, for each $i \in \{1, 2, \dots, 3n\}$, $i \equiv 1 \pmod{3}$, we can write

$$-\pi\bar{u}_{i+1}(1) + (\mathcal{L}^{i+1,i}\bar{u}_i)(1) + \sum_{\substack{j=1 \\ j \neq i}}^{3n} (\mathcal{K}^{i+1,j}\bar{u}_j)(1) + (\mathcal{M}^{i+1,i}\bar{u}_i)(1) = 0,$$

$$-\pi\bar{u}_{i+2}(0) + \sum_{j=1}^{3n} (\mathcal{K}^{i+2,j}\bar{u}_j)(0) = 0$$

and thus, being

$$(\mathcal{K}^{i+1,j}\bar{u}_j)(1) = (\mathcal{K}^{i+2,j}\bar{u}_j)(0), \quad j \neq i$$

and

$$(\mathcal{L}^{i+1,i}\bar{u}_i)(1) + (\mathcal{M}^{i+1,i}\bar{u}_i)(1) = (\mathcal{K}^{i+2,i}\bar{u}_i)(0),$$

it follows that $\bar{u}_{i+1}(1) = \bar{u}_{i+2}(0)$, i.e. $\bar{u} \in \tilde{\mathcal{X}}$.

Hence, since the operator $-\pi\mathcal{I} + \mathcal{L} + \mathcal{K}$ is invertible in $\tilde{\mathcal{X}}$ (see (32)), we can deduce that $\bar{u} = 0$ i.e. (67). Finally, if $\bar{g} \in \tilde{\mathcal{X}}$ then $\bar{u} = (-\pi\mathcal{I} + \mathcal{L} + \mathcal{K})^{-1}\bar{g} \in \tilde{\mathcal{X}}$. Indeed, since by (32) $(-\pi\mathcal{I} + \mathcal{L} + \mathcal{K})$ is invertible in $\tilde{\mathcal{X}}$, then there exists an array $\bar{\varphi} \in \tilde{\mathcal{X}} \subset \mathcal{X}$ such that $\bar{\varphi} = (-\pi\mathcal{I} + \mathcal{L} + \mathcal{K})^{-1}\bar{g}$. Hence, by (67) we can deduce $\bar{\varphi} \equiv \bar{u}$. \square

In order to prove Theorem 3.2 we need the following lemmas which can be proved by proceeding as in the proof of lemmas 2 and 3 in [14].

Lemma 4.1. *Let*

$$L(t, s) = -\frac{s \sin(\chi\pi)}{s^2 + 2ts \cos(\chi\pi) + t^2}, \quad t, s \in [0, 1],$$

for some $\chi \in \mathbb{R}$, $|\chi| < 1$, and let e_m be the functional defined as in (12). Then, for each $s \in (0, 1]$ one has

$$e_m(L(\cdot, s)) \leq \mathcal{C} \frac{r!}{m^r} \frac{1}{s^{r/2}},$$

where $r \in \mathbb{N}$ and $\mathcal{C} \neq \mathcal{C}(m)$.

Lemma 4.2. *Let \mathcal{X} be the space of functions defined in (33) and let*

$$\mathbb{P}^{3n} = \underbrace{\mathbb{P} \times \mathbb{P} \times \dots \times \mathbb{P}}_{3n \text{ times}},$$

where \mathbb{P} is the set of all polynomials on $[0, 1]$. Then the space

$$\tilde{\mathbb{P}}^{3n} = \mathbb{P}^{3n} \cap \mathcal{X} \quad (68)$$

is a dense subspace of \mathcal{X} .

Proof of Theorem 3.2. At first we note that the operators $\tilde{\mathcal{L}}_m$ map \mathcal{X} into \mathcal{X} . Indeed for any $i = 1, \dots, 3n$, $i \equiv 1 \pmod{3}$ and for any array of functions $\rho = (\rho_1, \dots, \rho_{3n})^T \in \mathcal{X}$, one has that

$$\tilde{\mathcal{L}}_m^i \varrho_i \in C([0, 1])^3 \quad \text{and} \quad (\tilde{\mathcal{L}}_m^i \varrho_i)(0) = (\mathcal{L}^i \varrho_i)(0),$$

with $\varrho_i = (\rho_i, \rho_{i+1}, \rho_{i+2})^T$.

Then $(\tilde{\mathcal{L}}_m \rho)(0) = (\mathcal{L} \rho)(0)$ and, consequently, $\tilde{\mathcal{L}}_m \rho \in \mathcal{X}$.

Now, in order to prove (44), we observe that for any $\rho = (\rho_1, \dots, \rho_{3n})^T \in \mathcal{X}$ such that $\|\rho\|_\infty \leq 1$ and for each $\varrho_i = (\rho_i, \rho_{i+1}, \rho_{i+2})^T$, we have

$$\begin{aligned} & \|\tilde{\mathcal{L}}_m \rho\|_\infty \\ &= \max_{\substack{i=1, \dots, 3n \\ i \equiv 1 \pmod{3}}} \left\| \tilde{\mathcal{L}}_m^i \varrho_i \right\|_\infty \\ &= \max_{\substack{i=1, \dots, 3n \\ i \equiv 1 \pmod{3}}} \max \left\{ \sup_{s \in [0, \frac{c}{m^2-2\epsilon}]} \|(\tilde{\mathcal{L}}_m^i \varrho_i)(s)\|_\infty, \sup_{s \in [\frac{c}{m^2-2\epsilon}, 1]} \|(\tilde{\mathcal{L}}_m^i \varrho_i)(s)\|_\infty \right\}. \end{aligned} \quad (69)$$

At this point, by repeating word by word the proof of Theorem 3 in [14], and taking into account that the number of the quadrature points μ_m is a linear function of m , one can prove that

$$\|\tilde{\mathcal{L}}_m \rho\|_\infty \leq \chi \pi + \mathcal{C} \frac{r!}{m^{r\epsilon}}, \quad r \in \mathbb{N}, \quad (70)$$

where $\chi = \max\{|\chi_k|, k = 1, \dots, n\} < 1$ with χ_k defined in (7). From (70) we deduce

$$\limsup_{m \rightarrow \infty} \|\tilde{\mathcal{L}}_m \rho\|_\infty \leq \chi \pi \quad (71)$$

and, consequently, (44). Now we prove (45). To do this, we observe that from (71) it follows that the operators $\tilde{\mathcal{L}}_m : \mathcal{X} \rightarrow \mathcal{X}$ are uniformly bounded with respect to m , i.e.

$$\sup_m \|\tilde{\mathcal{L}}_m\| < \infty. \quad (72)$$

Moreover, by proceeding analogously to the proof of Theorem 3 in [14], we can prove that for any $\rho = (\rho_1, \dots, \rho_{3n})^T \in \tilde{\mathbb{P}}^{3n}$, setting $\varrho_i = (\rho_i, \rho_{i+1}, \rho_{i+2})^T$, one has

$$\lim_{m \rightarrow \infty} \|(\tilde{\mathcal{L}}_m^i - \mathcal{L}^i)\varrho_i\|_\infty = 0, \quad \forall i \in \{1, \dots, 3n\}, \quad i \equiv 1 \pmod{3}. \quad (73)$$

Hence, taking into account Lemma 4.2, we can deduce (45) by applying the Banach-Steinhaus theorem (see, for instance, [3, p. 517]). The proof is complete. \square

Proof of Theorem 3.3. At first let us note that the operators \mathcal{K}_m map \mathcal{X} into \mathcal{X} and the set $\{\mathcal{K}_m\}_m$ is collectively compact if so is the set $\{\mathcal{K}_m^{i,j}\}_m$ for any $i, j \equiv 1, 2 \pmod{3}$ with $|i - j| \neq 1$ or $i \equiv 0 \pmod{3}$ and the set $\{\mathcal{M}_m^{i,j}\}_m$ for any $i, j \equiv 1, 2 \pmod{3}$ with $|i - j| = 1$. Moreover, by definition, it results that $\forall \rho = (\rho_1, \dots, \rho_{3n}) \in \mathcal{X}$, if

$$\lim_{m \rightarrow \infty} \|(\mathcal{K}_m^{i,j} - \mathcal{K}^{i,j})\rho_j\|_\infty = 0 \quad (74)$$

when $i, j \equiv 1, 2 \pmod{3}$ with $|i - j| \neq 1$ or $i \equiv 0 \pmod{3}$, and

$$\lim_{m \rightarrow \infty} \|(\mathcal{M}_m^{i,j} - \mathcal{M}^{i,j})\rho_j\|_\infty = 0 \quad (75)$$

when $i, j \equiv 1, 2 \pmod{3}$ with $|i - j| = 1$, then

$$\lim_{m \rightarrow \infty} \|(\mathcal{K}_m - \mathcal{K})\rho\|_\infty = 0.$$

Now, the limit conditions (74) (see, for instance, [3]) and (75) (see, for instance, [1, 2]) can be immediately deduced taking into account the definitions (36), (37) and (38) of the finite rank operators $\mathcal{K}_m^{i,j}$ and $\mathcal{M}_m^{i,j}$, the convergence of the Radau quadrature rule on the set $C([0, 1])$, the positivity of its weights and the properties of the kernels $K^{i,j}(\cdot, s)$ and $M^{i,j}(\cdot, s)$ (see Section 3). It also follows that the sets $\{\mathcal{K}_m^{i,j}\}_m$ (see, for instance, [19, Theorem 12.8]) and $\{\mathcal{M}_m^{i,j}\}_m$ (see [2, Theorem 5.1]), with i and j as above, are collectively compact and the proof is complete. \square

Proof of Theorem 3.4. From Theorem 3.2 we can deduce that the operators $-\pi\mathcal{I} + \tilde{\mathcal{L}}_m : \mathcal{X} \rightarrow \mathcal{X}$ are bounded and pointwise convergent to $-\pi\mathcal{I} + \mathcal{L}$. Moreover, in virtue of the geometric series theorem, it follows that for sufficiently large m , say $m \geq m_0$, the operators $(-\pi\mathcal{I} + \tilde{\mathcal{L}}_m)^{-1} : \mathcal{X} \rightarrow \mathcal{X}$ exist and are uniformly bounded with

$$\|(-\pi\mathcal{I} + \tilde{\mathcal{L}}_m)^{-1}\| \leq \frac{1}{\pi - \sup_{n \geq m_0} \|\tilde{\mathcal{L}}_n\|}.$$

Consequently, taking also into account Theorem 3.3, it results (see, for instance, Theorem 10.8 and Problem 10.3 in [19]) that for sufficiently large m the operators

$$(-\pi\mathcal{I} + \tilde{\mathcal{L}}_m + \mathcal{K}_m)^{-1} : \mathcal{X} \rightarrow \mathcal{X}$$

exist and are uniformly bounded, i.e. the method is stable. Now, by using the same arguments as in the proof of theorems 5 and 6 in [14], one can assert (50) and finally show that the error estimate (52) holds true. \square

In order to prove Theorem 3.5, we recall the definition of the error of best polynomial approximation in uniform norm for a function $f \in C([0, 1])$

$$E_m(f)_\infty = \inf_{P_m \in \mathbb{P}_m} \|(f - P_m)\|_\infty,$$

and the following error estimate for the Gauss-Legendre quadrature formula (9)

$$|e_m(f)| \leq \mathcal{C} E_{2m-1}(f), \quad \mathcal{C} \neq \mathcal{C}(m, f). \quad (76)$$

Moreover we mention that for $f, g \in C([0, 1])$ it results

$$E_{2m-1}(fg)_\infty \leq 2\|f\|_\infty E_{\lfloor \frac{2m-1}{2} \rfloor}(g)_\infty + E_{\lfloor \frac{2m-1}{2} \rfloor}(f)_\infty \|g\|_\infty. \quad (77)$$

Proof of Theorem 3.5. Since

$$\|(\bar{g} - \bar{g}_M)(s)\|_\infty = \max_{i=1, \dots, 3n} |\bar{g}_i(s) - \bar{g}_{M,i}(s)|,$$

in order to prove (59), we are going to estimate the i -th term $|\bar{g}_i(s) - \bar{g}_{M,i}(s)|$. By (54), (56) and (58), it results

$$|\bar{g}_i(s) - \bar{g}_{M,i}(s)| \leq \sum_{\substack{k=1 \\ k \neq \ell}}^n \left| \int_0^1 \phi_k(t) \log |\tilde{\sigma}_k(s_i) - \tilde{\sigma}_\ell(t)| dt \right|$$

$$\begin{aligned}
& - \sum_{h=1}^M \lambda_{M,h}^L \phi_k(x_{M,h}^L) \log |\tilde{\sigma}_k(s_i) - \sigma_\ell(x_{M,h}^L)| \Big| \\
& + \left| \int_0^1 \phi_\ell(t) \log |t - s_i| dt - \sum_{h=1}^M \lambda_{M,h}^L \phi_\ell(x_{M,h}^L) \sum_{\nu=0}^{M-1} c_\nu(s_i) p_\nu(x_{M,h}^L) \right| \\
& + \left| \int_0^1 \phi_\ell(t) \delta_\ell(t, s_i) dt - \sum_{h=1}^M \lambda_{M,h}^L \phi_\ell(x_{M,h}^L) \delta_\ell(x_{M,h}^L, s_i) \right| \\
& =: \sum_{\substack{k=1 \\ k \neq \ell}}^n |A_{k,\ell}(s_i)| + |B_\ell(s_i)| + |C_\ell(s_i)|. \tag{78}
\end{aligned}$$

Now let us consider $A_{k,\ell}(s_i)$, for k fixed. By (76) and (77), one can write

$$\begin{aligned}
|A_{k,\ell}(s_i)| & \leq \mathcal{C} E_{2M-1}(\phi_k \log |\tilde{\sigma}_k(s_i) - \tilde{\sigma}_\ell(\cdot)|)_\infty \\
& \leq \mathcal{C} \left(E_{\lceil \frac{2M-1}{2} \rceil}(\phi_k)_\infty \| \log |\tilde{\sigma}_k(s_i) - \tilde{\sigma}_\ell(\cdot)| \|_\infty \right. \\
& \quad \left. + 2 \|\phi_k\|_\infty E_{\lceil \frac{2M-1}{2} \rceil}(\log |\tilde{\sigma}_k(s_i) - \tilde{\sigma}_\ell(\cdot)|)_\infty \right) \\
& \leq \frac{\mathcal{C}}{M}, \tag{79}
\end{aligned}$$

being, by the assumptions, $\phi_k \in C^1$ and $\log |\tilde{\sigma}_k(s_i) - \tilde{\sigma}_\ell(\cdot)| \in C^2$ for each $s_i \in [0, 1]$. Using the same arguments, one has that

$$|C_\ell(s_i)| \leq \frac{\mathcal{C}}{M} \tag{80}$$

being, under the hypotheses, $\phi_\ell \in C^1$ and $\delta_\ell(\cdot, s_i) \in C^1$ for each $s_i \in [0, 1]$ and for each $\ell = 1, \dots, n$. Finally, by applying [24, (3.11)], it results

$$|B_\ell(s_i)| \leq \frac{\mathcal{C}}{M}. \tag{81}$$

Hence, by using (79), (80) and (81) in (78), the thesis follows. \square

Proof of Theorem 3.6. Estimate (64) can be proved by proceeding as in the proof of Theorem 7 in [14], taking also into account that the number of the involved quadrature nodes is a linear function of m . Concerning inequality (65), we note that, by definition, it results

$$|u(x, y) - u_{m,N}(x, y)| \leq |u(x, y) - u_m(x, y)| + |R_N(x, y)|, \tag{82}$$

where

$$R_N(x, y) = \sum_{k=1}^n \left(\int_0^1 \phi_k(t) \log |(\tilde{\xi}_k(t), \tilde{\eta}_k(t)) - (x, y)| dt - \sum_{h=1}^N \lambda_{M,h}^L \phi_k(x_{M,h}^L) \log |(\tilde{\xi}_k(x_{M,h}^L), \tilde{\eta}_k(x_{M,h}^L)) - (x, y)| \right) \quad (83)$$

$$=: \sum_{k=1}^n R_{N,k}(x, y). \quad (84)$$

Then, taking into account (11), we can write

$$\begin{aligned} |R_{N,k}(x, y)| &\leq \frac{\mathcal{C}}{N} E_{2N-2}((\phi_k(\cdot) \log |(\tilde{\xi}_k(\cdot), \tilde{\eta}_k(\cdot)) - (x, y)|)')_{\varphi,1} \\ &\leq \frac{\mathcal{C}}{N} \|(\phi_k(\cdot) \log |(\tilde{\xi}_k(\cdot), \tilde{\eta}_k(\cdot)) - (x, y)|)'\varphi\|_1 \\ &\leq \frac{\mathcal{C}}{N} \int_0^1 \sum_{j=0}^1 (\log |(\tilde{\xi}_k(t), \tilde{\eta}_k(t)) - (x, y)|)^{(j)} \phi_k^{(1-j)}(t) t^{\frac{1}{2}} dt \\ &\leq \frac{\mathcal{C}}{N} \sum_{j=0}^1 \|(\log |(\tilde{\xi}_k(\cdot), \tilde{\eta}_k(\cdot)) - (x, y)|)^{(j)}\|_{\infty} \\ &\leq \frac{\mathcal{C}}{dN}, \end{aligned} \quad (85)$$

where $d = \min_{k=1, \dots, n} d_k$ with $d_k = \min_{0 \leq t \leq 1} |(x, y) - (\tilde{\xi}_k(t), \tilde{\eta}_k(t))|$. Hence, by using (64) and (85) in (82), we get (65). \square

5. Numerical tests

In this section we apply the method described in Section 3 for the numerical solution of some examples of the exterior Neumann problem on planar domains with corners.

In most tests, in order to give the boundary condition f , we choose a test harmonic function u and we present the absolute error

$$\varepsilon_{m,N}(x, y) = |u(x, y) - u_{m,N}(x, y)|$$

at the point $(x, y) \in \mathbb{R}^2 \setminus \overline{D}$ where $u_{m,N}$ is as in (63). Moreover, we also analyze the condition number in infinity norm of the matrix A_m of the linear

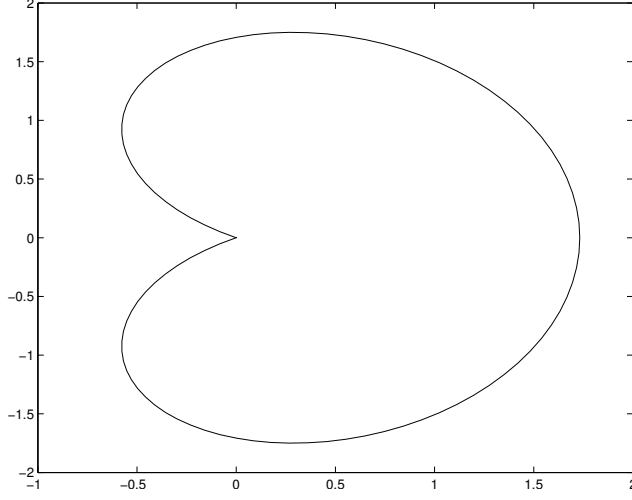


Figure 1: The contour Σ in Example 5.1 with $\phi = \frac{5}{3}\pi$

system (49).

All the numerical results are obtained by approximating the right-hand sides by using (58) with $M = \frac{\nu_m}{2}$.

Example 5.1. *Let us consider a family of “heart-shaped” domains (see Figure 1) bounded by the curves*

$$\sigma(t) = \begin{pmatrix} \cos(1 + \frac{\phi}{\pi})\pi t - \sin(1 + \frac{\phi}{\pi})\pi t \\ \sin(1 + \frac{\phi}{\pi})\pi t + \cos(1 + \frac{\phi}{\pi})\pi t \end{pmatrix} \begin{pmatrix} \tan \frac{\phi}{2} \\ 1 \end{pmatrix} - \begin{pmatrix} \tan \frac{\phi}{2} \\ \cos \pi t \end{pmatrix}, \quad t \in [0, 1],$$

where $\phi \in (\pi, 2\pi)$ is the interior angle of the single outward-pointing corner $P_1 = (0, 0)$. For this test we choose boundary data corresponding to the exact solution

$$u(P) = \log |P - Q_1| - \log |P - Q_2|, \quad Q_1 = (0.5, 0), \quad Q_2 = (0.2, 0).$$

Table 1 reports the results obtained by applying our method for $\phi = \frac{5}{3}\pi$ and $N = \frac{\nu_m}{2}$ while Figure 2 shows the condition number in infinity norm of the matrix A_m as a function of the interior angle ϕ , confirming that the estimate (50) holds true whatever the angle at the corner point.

Example 5.2. *Let us consider the family of the “heart-shaped” domains introduced in the previous example but now let us choose the Neumann data*

Table 1: Numerical results for Example 5.1 with $\phi = \frac{5}{3}\pi$, $c = 300$ and $\epsilon = 10^{-3}$

μ_m	ν_m	$\varepsilon_{m,N}(-0.1, 0)$	$\varepsilon_{m,N}(3, 3)$	$\varepsilon_{m,N}(-40, -50)$	$\varepsilon_{m,N}(100, -100)$	$\text{cond}(A_m)$
8	32	6.62e-03	2.35e-05	4.52e-05	5.9e-05	133.5
16	64	6.95e-03	1.89e-04	1.12e-05	5.3e-06	25.86
32	128	6.78e-04	1.81e-05	1.05e-06	5.3e-07	18.37
64	256	1.18e-05	3.19e-07	1.86e-08	9.2e-09	18.32
128	512	2.29e-06	6.10e-08	3.55e-09	1.8e-09	18.32

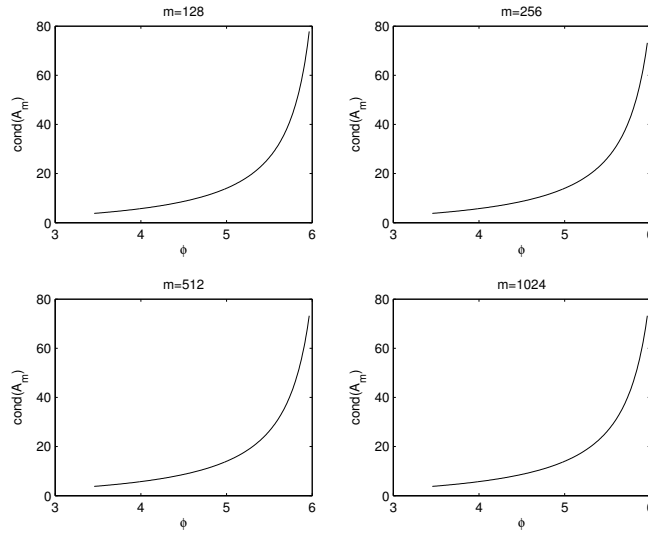


Figure 2: Condition Numbers for Example 5.1 with $c = 300$ and $\epsilon = 10^{-3}$

$f(\sigma(t)) = (1 - 3t^2)/|\sigma'(t)|$, $t \in [0, 1]$ for which the exact solution of (1) is not known. Note that the asymptotic relations (34) and (35) are sharp. In this case we perform the absolute errors

$$\bar{\varepsilon}_{m,N}(x, y) = |u_{1024,512}(x, y) - u_{m,N}(x, y)|$$

at the point $(x, y) \in \mathbb{R}^2 \setminus \overline{D}$ where $u_{m,N}$ is as in (63). Table 2 shows the numerical results we have with $\phi = \frac{5}{3}\pi$ and $N = \frac{\nu_m}{2}$.

Example 5.3. Let us consider a family of “teardrop” domains (see Figure 3) bounded by the curves parameterized by

$$\sigma(t) = \left(2 \sin \pi t, -\tan \frac{\phi}{2} \sin 2\pi t \right), \quad t \in [0, 1],$$

Table 2: Numerical results for Example 5.2 with $\phi = \frac{5}{3}\pi$, $c = 300$ and $\epsilon = 10^{-3}$

μ_m	ν_m	$\bar{\varepsilon}_{m,N}(-0.1, 0)$	$\bar{\varepsilon}_{m,N}(3, 3)$	$\bar{\varepsilon}_{m,N}(-40, -50)$	$\bar{\varepsilon}_{m,N}(100, -100)$
8	32	3.54e-04	1.33e-05	7.36e-07	8.68e-07
16	64	9.23e-04	2.42e-05	1.41e-06	7.05e-07
32	128	8.70e-05	2.31e-06	1.35e-07	6.86e-08
64	256	2.04e-06	5.51e-08	3.21e-09	1.60e-09
128	512	3.23e-07	8.74e-09	5.11e-10	2.53e-10

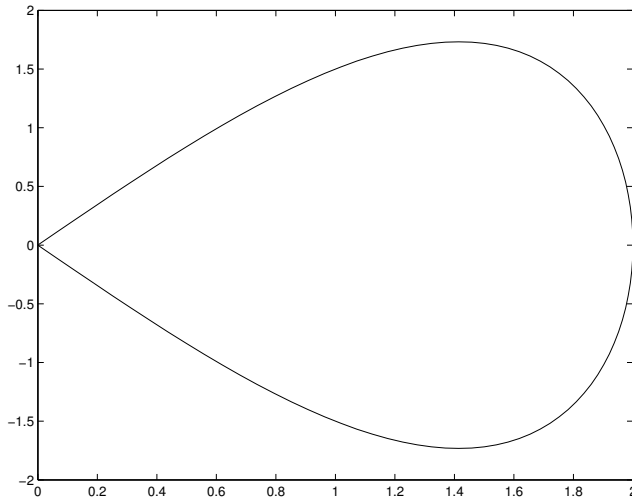


Figure 3: The contour Σ in Example 5.3 with $\phi = \frac{2}{3}\pi$

where $\phi \in (0, \pi)$ is the interior angle of the single outward-pointing corner $P_1 = (0, 0)$ and let us choose the boundary data f as the normal derivative of the following function

$$u(x, y) = \arctan\left(\frac{y - 0.2}{x - 0.8}\right) - \arctan\left(\frac{y}{x - 0.8}\right).$$

The numerical results, reported in Table 3 and Table 4, are obtained for $\phi = \frac{2}{3}\pi$ and $N = \frac{\nu_m}{2}$. In Table 4 the estimated orders of convergence at the point $(x, y) \in \mathbb{R}^2 \setminus \bar{D}$ defined as

$$EOC(x, y) = \frac{\log(\varepsilon_{m,N}(x, y)/\varepsilon_{2m,N}(x, y))}{\log 2}$$

are shown. As we can see, the method converges and the estimated values

$EOC(x, y)$ are even better than the predicted convergence orders given by estimate (65). Moreover, Figure 4 confirms that the sequence $\{\text{cond}(A_m)\}_{m \geq m_0}$ is uniformly bounded with respect to m , according to the theoretical estimate (50).

Table 3: Numerical results for Example 5.3 with $\phi = \frac{2}{3}\pi$, $c = 100$ and $\epsilon = 10^{-3}$

μ_m	ν_m	$\varepsilon_{m,N}(-0.01, 0)$	$\varepsilon_{m,N}(3, 3)$	$\varepsilon_{m,N}(-40, -50)$	$\varepsilon_{m,N}(100, -100)$	$\text{cond}(A_m)$
8	32	4.15e-03	6.39e-04	1.47e-03	1.80e-03	6.67
16	64	4.80e-05	8.81e-06	4.34e-06	5.57e-06	4.49
32	128	3.70e-06	2.54e-07	1.98e-08	8.05e-09	4.16
64	256	1.19e-06	1.03e-08	8.14e-10	3.62e-10	4.16
128	512	3.18e-07	2.92e-09	2.29e-10	1.01e-10	4.17

Table 4: Estimated order of convergence for Example 5.3

m	$EOC(-0.01, 0)$	$EOC(3, 3)$	$EOC(-40, -50)$	$EOC(100, -100)$
32	6.43	6.18	8.41	8.33
64	3.69	5.11	7.77	9.43
128	1.62	4.62	4.60	4.47
256	1.91	1.82	1.82	1.83

Example 5.4. Let us consider a family of “boomerang” domains (see Figure 5) having as boundaries the following curves

$$\sigma(t) = \left(\frac{2}{3} \sin 3\pi t, -\tan \frac{\phi}{2} \sin 2\pi t \right), \quad t \in [0, 1],$$

where $\phi \in (\pi, 2\pi)$ is the interior angle of the single inward-pointing corner $P_1 = (0, 0)$. Table 5 shows the numerical results obtained in the case where the boundary data f is the normal derivative of the function

$$u(P) = \log |P - Q_1| - \log |P - Q_2|, \quad Q_1 = (-0.1, 0), \quad Q_2 = (-0.2, 0),$$

the interior angle at P_1 is $\phi = \frac{3}{2}\pi$ and $N = \nu_m$.

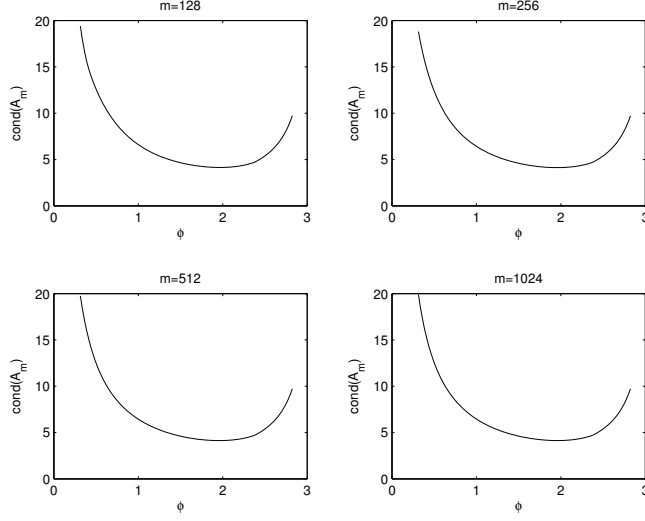


Figure 4: Condition Numbers for Example 5.3 with $c = 100$ and $\epsilon = 10^{-3}$

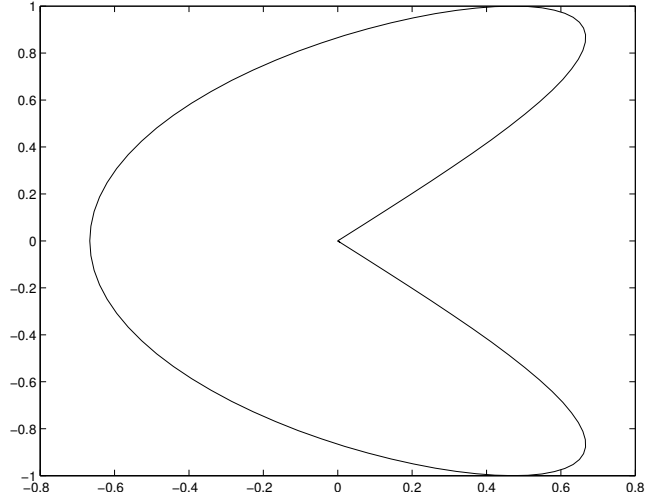


Figure 5: The contour Σ in Example 5.4 with $\phi = \frac{3}{2}\pi$

Example 5.5. Let D be the polygonal domain represented in Figure 7 with vertices $(-5/4, -3/4)$, $(3/4, -3/4)$ and $(3/4, 5/4)$ and apply the method described in Section 3 in the case when the exact solution of (1) is the following

Table 5: Numerical results for Example 5.4 with $\phi = \frac{3}{2}\pi$, $c = 100$ and $\epsilon = 10^{-3}$

μ_m	ν_m	$\varepsilon_{m,N}(0.2, 0)$	$\varepsilon_{m,N}(3, 3)$	$\varepsilon_{m,N}(-40, -50)$	$\varepsilon_{m,N}(100, -100)$	$\text{cond}(A_m)$
8	32	7.22e-03	2.66e-04	7.41e-06	3.48e-05	19.13
16	64	3.34e-04	1.62e-05	9.59e-07	4.87e-07	16.92
32	128	8.51e-05	4.05e-06	2.39e-07	1.21e-07	16.92
64	256	1.95e-05	9.34e-07	5.54e-08	2.80e-08	16.93
128	512	4.64e-06	2.22e-07	1.31e-08	6.67e-09	16.93

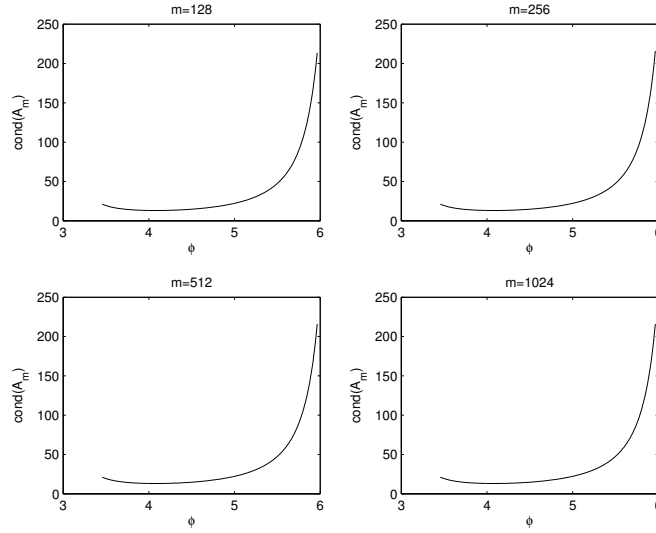


Figure 6: Condition Numbers for Example 5.4 with $c = 300$ and $\epsilon = 10^{-3}$

harmonic function

$$u(x, y) = \frac{x^2 - y^2}{(x^2 + y^2)^2}.$$

Table 6 contains the numerical results obtained with $N = \frac{\nu_m}{2}$.

Remarks

The numerical results, shown in this section, confirm the theoretical ones stated in Section 3. We can note that, according to estimate (65), for any fixed m , the error $\varepsilon_{m,N}(x, y)$, becomes smaller and smaller as well as the distance of the exterior point (x, y) from the boundary is larger and larger. Moreover, the results put in evidence that, as stated in (50), the sequence $\{\text{cond}(A_m)\}_{m \geq m_0}$ is uniformly bounded with respect to m .

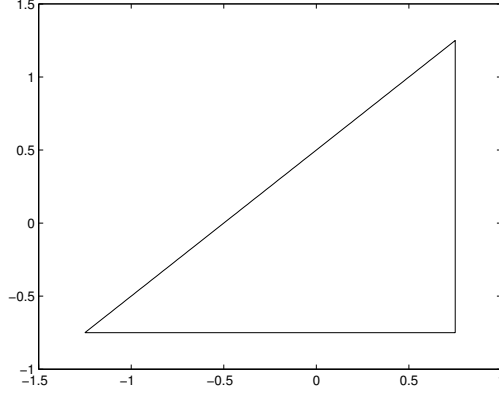


Figure 7: The contour Σ in Example 5.5

Table 6: Numerical results for Example 5.5 with $c = 100$ and $\epsilon = 10^{-6}$

μ_m	ν_m	$\varepsilon_{m,N}(-1.5, 1.5)$	$\varepsilon_{m,N}(2, 2)$	$\varepsilon_{m,N}(10, 20)$	$\varepsilon_{m,N}(100, 100)$	$\text{cond}(A_m)$
8	32	7.11e-04	1.33e-02	1.79e-03	2.93e-04	166.39
16	64	9.83e-04	2.78e-04	6.51e-05	6.70e-06	66.18
32	128	1.41e-04	7.74e-06	6.94e-06	5.27e-07	20.40
64	256	2.18e-06	3.38e-07	1.24e-07	1.12e-08	9.11
128	512	6.93e-09	1.20e-09	4.16e-10	3.90e-11	8.81

The computational cost of the proposed procedure of course grows with the number of corners of the domain. However, when the requested precision is not too high (as it is usual in the applications), the dimension of the linear system (49) is kept down. Anyway, such system is still well conditioned also when its dimension is larger, whatever the interior angles at the corner points.

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